Vibrant Technology, Inc.

ME'scopeVES Application Note #5

Impact Testing Question

INTRODUCTION

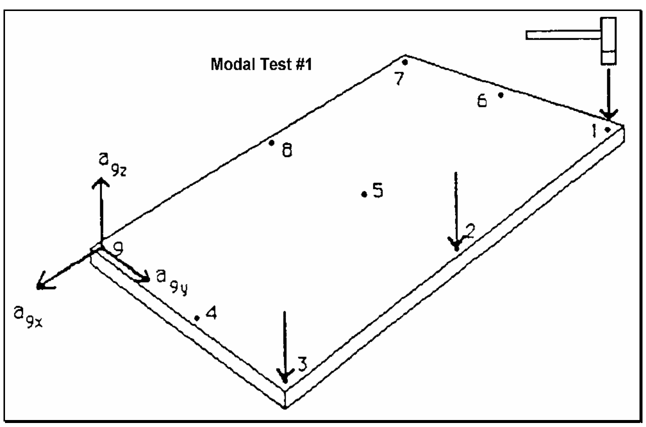
There are two ways to perform a modal test on a structure using an impact hammer and a tri-axial accelerometer. One test is called a *roving impact test* and the other is called a *roving accelerometer test.*

THE QUESTION

If both of these methods are applied to a structure with nine measurement points of interest, do the two test methods yield the same modal information?

Roving Impact Test

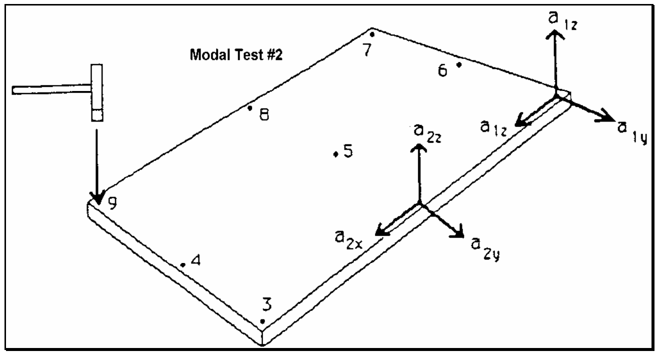
In a roving impact test, Frequency Response Function (FRF) measurements are made by attaching the tri-axial accelerometer at a fixed point in the structure (say Point **9**), and impacting the it at points **1** through **9** in the **Z** direction. Since three acceleration outputs are simultaneously measured for each force input, a total of **27** FRF measurements are gathered.



Roving Impact Test.

Roving Accelerometer Test

In a roving accelerometer test, FRF measurements are made by consistently impacting the structure *at the same point and in the same direction* (say **9Z**, for example). The tri-axial accelerometer is attached to point **1** for the first measurement and is then moved to point **2**, and so on. Again, a total of **27** FRF measurements are collected.



Roving Accelerometer Test.

Even though the structure illustrated is a flat plate, this question really applies to testing *any* structure. The Z direction can be any direction you choose, and it can be *a different direction at each point*.

THE ANSWER

The two tests are not equivalent. The *roving impact test* will provide mode shapes with nine degrees-of-freedom (DOFs). Motions at the nine test points in the **Z** direction will be contained in each mode shape.

In contrast, the *roving accelerometer test* will provide mode shapes with 27 DOFs. The mode shapes will exhibit motion in the **X**, **Y** and **Z** directions at each of the nine test point locations.

On the surface, the *roving impact test* appears to offer little merit. However, this is *not* true. This test provides *three measured estimates* for each mode shape component; the *roving accelerometer test* only provides one.

Should the structure prove difficult to analyze, the data measured by the *roving impact test* can be fitted using *multiple reference* methods that can separate *repeated roots* or deal with structures that have groups of *local modes*, shapes physically segregated by direction or location. Data from the *roving accelerometer test* cannot be used in this manner.

In short, the roving tri-axial accelerometer test is a *single reference* test measuring 3 DOFs at each point. The roving impact test using a triaxial accelerometer is a *three reference test* measuring 1 DOF at each point.

A MORE DETAILED EXPLANATION

An experimental modal analysis characterizes dynamics at **N** degrees-of-freedom (DOFs), each a *motion in a specific direction at a particular point on the structure*. Hence, there are $\mathbf{N^2}$ possible FRFs that could be measured between pairs of the **N** DOFs. These measurements can be arranged in an **N** by **N** square matrix, $\mathbf{H_{ij}}$, where $\mathbf{H_{ij}}$ = displacement at DOF, **i**, per unit force applied to DOF, **j**, as a function of frequency.

In general, it is unnecessary to measure all $\mathbf{N^2}$ FRFs. Most structures can be completely characterized by measuring just *a single row* or *a single column* of the FRF matrix. That is, only **N** FRFs normally need to be measured. The reason for this is explained later.

Modes of vibration are basically defined by *three* parameters:

1. Modal frequency
2. Modal damping
3. Mode Shape

When the modes are used to predict the effects of a *Structural Modification* or combined with other modal models through *Substructuring*, a fourth modal parameter, termed *modal mass*, is also needed.

The test will capture data needed to identify many modes (say **M** of them) from the one set of measurements. *Every* FRF will exhibit characteristics reflecting the frequency and damping of mode shapes. The set of **N** FRFs is needed to identify mode shapes. Each *mode shape vector* has **N** elements, one for each DOF measured.

Often, a structure will exhibit strong *spatial separation* of modes, where *a single DOF cannot be found that reflects all* **M** *modes*. These modes are referred to as *local* modes. In this instance, either a (fixed) reference DOF must be chosen which exhibits all of the modes, or two or more columns or rows must be measured to gather enough data for a complete modal analysis.

More than **N** FRFs may also be required from structures with strong *spatial symmetry*. Such structures often exhibit *repeated roots* (two or modes at the *same frequency*, each with a *different mode shape*). Again, two or more rows or columns from the $\mathbf{H_{ij}}$ matrix must be measured to complete the modal analysis.

Let's examine the analytical formula for the FRF matrix to see how these parameters are contained in a set of measurements. The FRF matrix can be written in *partial fraction expansion* form as a *summation of pairs of terms*, each pair containing the contribution of a single mode.

$$\left[H_{ij}(j\omega)\right] = \sum_{k=1}^{M} \frac{1}{2j}\left[\frac{\left[R_{ij}(k)\right]}{(j\omega - p_k)} - \frac{\left[R_{ij}^*(k)\right]}{(j\omega - p_k^*)}\right] \quad (1)$$

where:

$H_{ij}(j\omega)$ = the element in the **i**th row and **j**th column of the FRF matrix, representing displacement of DOF, **i**, resulting from a force applied to DOF, **j**, with typical units of m/N or in/lb.

p_k = the **k**th *pole* of the structure.
$= -\sigma_k + j\omega_k$ (rad/sec)

ω_k = damped natural frequency of the **k**th mode.

σ_k = damping of **k**th mode. (rad/sec)

$R_{ij}(k)$= Residue matrix element relating DOF's **i** and **j** in the **k**th mode. (m/Ns or in/lb-sec)

ω = the forcing frequency (rad/sec)

$j = \sqrt{-1}$ and * indicates complex conjugation.

Each FRF is the sum of **2M** terms. Each term contains a *residue* divided by a *pole*. The *denominator* of all FRFs is the *same*; each term contains the same pair of poles for each mode, **k**.

The FRFs *differ only by numerators*. Each numerator contains a specific residue, $R_{ij}(k)$, which is dependent upon the response (**i**) and excitation (**j**) DOFs and each mode, **k**.

Residues are *physical properties* of a structure and have the units of *velocity/force* (m/Ns or in/lb-sec). A *residue matrix* is an **N** by **N** collection of the residues associated with a single mode. When FRFs are measured, we actually measure the elements of **M** residue matrices *simultaneously*, one for each mode encountered.

Normally, the FRF matrix is assumed to be *symmetric*. This follows from the assumption of *structural reciprocity*. Consequently, the residue matrix is also symmetric.

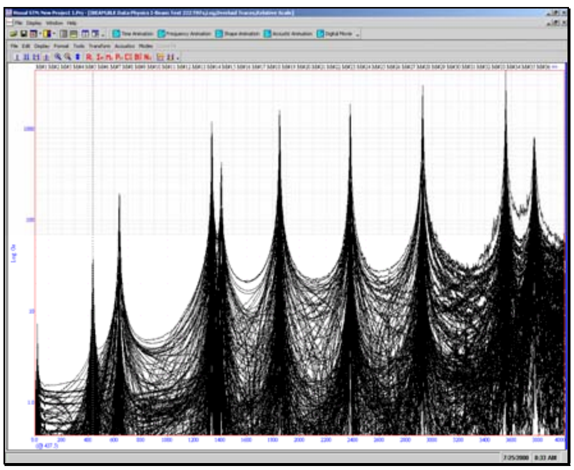
In *single-reference testing*, we measure a *single* row or column of each residue matrix. This is (normally) sufficient information to represent the entire residue matrix.

It is shown later that each residue matrix contains one *mode shape* in each of its rows and columns. Each residue matrix element is the product of two mode shape elements divided by the mode's *modal mass*.

GLOBAL MODES

Since the *same poles* are contained in *all* FRF measurements, we can (in principle) identify all **M** modal frequencies and damping from a *single* measurement. We gather a larger number of measurements, **N**, solely to collect the residues needed to determine the *mode shapes*.

The *Global Properties* (frequency and damping), contained in the poles are present in every measured FRF. However, they are easier to detect from some measurements than from others. In general, a particular *pole* is best identified from measurements that have a large resonance peak (large associated *residue*) for that pole.

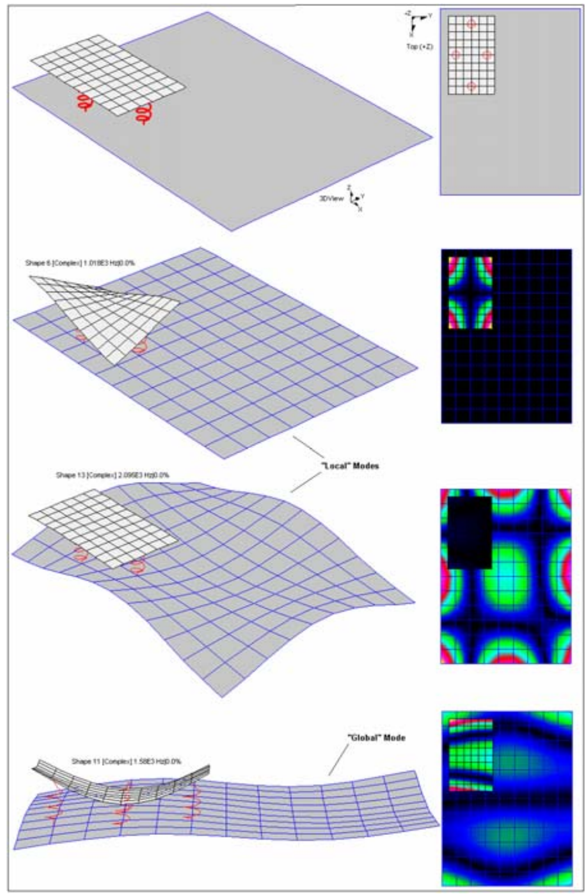


Overlaid FRF Magnitudes show common poles.

The figure above shows the magnitudes of a number of FRFs overlaid on one another. Notice that the resonance peaks all appear at the same frequencies. This shows that the modal frequencies of all of the modes are the same no matter where the FRFs are measured on the structure. This shows the *global* nature of modal frequencies.

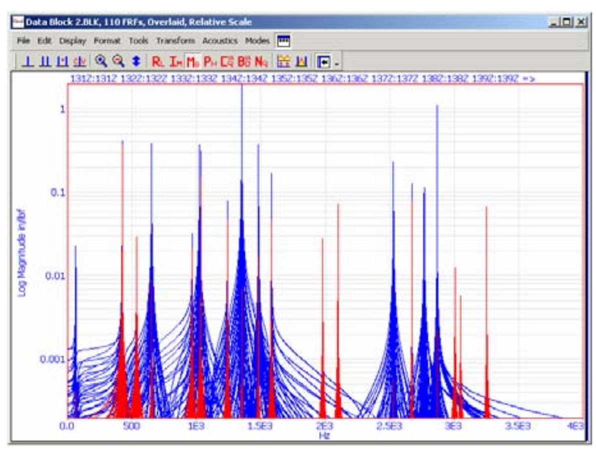
LOCAL MODES

In structures comprised of multiple components, it is not uncommon to encounter a *local mode* phenomenon. The combination of two plates joined by four springs shown at right illustrates such a situation. The small plate exhibits some modes in which the base plate does *not* participate and vice versa. Driving-point FRFs gathered on the small plate (red) show some different resonances from those gathered on the base plate (blue).



Local mode behavior in a two-plate structure.

Local modes are often the result of deliberate design intent. In this instance, the four mounting springs were placed on the centerlines of the small plate. These locations are Z-axis *nodal points* for many of the small plate's modes, including *all* torsional modes.



Driving-point FRFs show local mode behavior.

As another example, automotive exhaust systems are attached to the car chassis by hangers deliberately placed at *nodal points* of the exhaust piping.

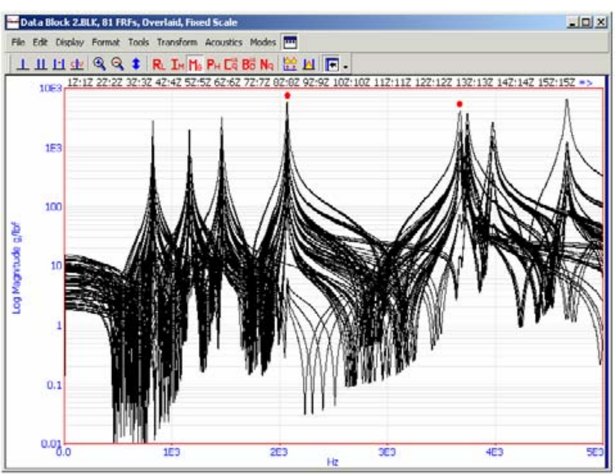
MULIPLE REFERENCE TEST

When *local modes* are encountered, a *multiple reference* analysis is required so that no modes will be missed. As an example, using a single reference on the *base plate* of the previously illustrated two-plate structure would fail to capture the *local modes* of the small mounted plate.

Using a single reference on the small attached plate would correct this, but then some local modes of the base plate would be missed. Either *single reference* test would capture the *global modes* common to both plates, but miss some of the local modes.

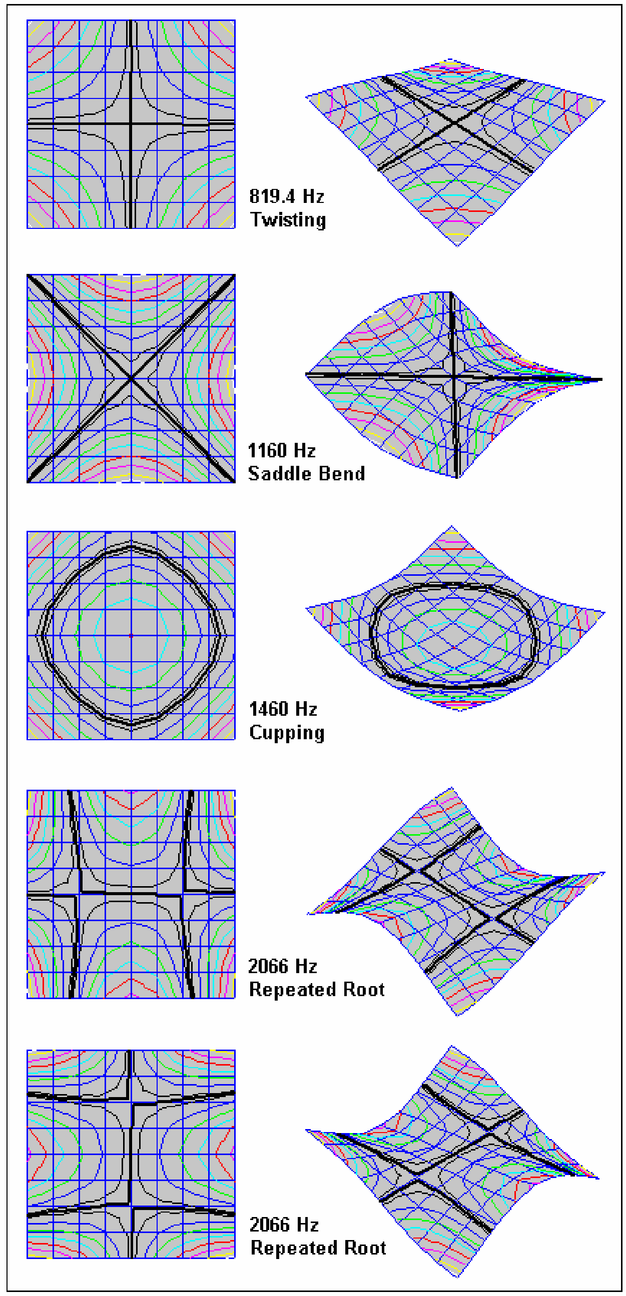
Multiple reference testing and analysis is also required when the structure is very simple and geometrically symmetrical. Consider the first five modes of a free-free square plate illustrated at right. While the first three modes are *unique roots* amenable to *single reference testing*, the fourth and fifth modes are a repeated-root pair.

A square plate will exhibit many *pairs* of repeated roots. The mode shape for one of these modes is always identical to the shape for the other, but is rotated by 90° about the axis of symmetry (**Z** in this instance). Repeated roots of *higher multiplicity* than two are also possiblebut are rarely encountered.



Driving-point FRFs from square plate.

Measured FRFs give no direct indication of a repeated root. The square plate measurements above include two pairs of repeated poles (at red dots) and six distinct roots.



A free-free square plate exhibits repeated roots.

When repeated poles are encountered, special processing of multiple rows or columns of FRFs is required to separate the *different* shapes that occur at the same frequency.

Suggestion: you can use two accelerometers and the Coherence Function to test your structure for a suspected repeated root. Measure the coherence between accelerometers at two different locations while impacting the structure at many different locations. Read the Coherence at the resonance frequency in question. A value greater than 0.9 indicates a *unique* pole; a low value indicates a *repeated root*.

MORE ABOUT MODE SHAPES

The measured motion of a structure at **N** DOFs may be listed in a vector of **N** elements, $\{x\}$ such that:

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ : \\ x_i \\ x_N \end{Bmatrix} \quad (2)$$

Where $\mathbf{x_i}$ is the displacement of the $\mathbf{i}^{th}$ degree-of-freedom.

This vector is also called an *Operating Deflection Shape* (ODS) or simply a deflection shape.

The first principle of Modal analysis is that *any* measured overall motion vector can be expressed as *a linear combination of* vectors having *known and fixed* patterns called *mode shapes*. Each mode shape can be written as a vector with **N** elements. For example, the $\mathbf{k}^{th}$ mode shape, $\{u_k\}$, may be stated:

$$\{u_k\} = \begin{Bmatrix} u_{1k} \\ u_{2k} \\ : \\ u_{ik} \\ u_{Nk} \end{Bmatrix} \quad (3)$$

Where u_{ik} is the motion of the $\mathbf{i}^{th}$ DOF in the $\mathbf{k}^{th}$ mode.

M mode shape vectors may be arranged as the *columns* of a *mode shape matrix*,[U] , of the form:

$$[U] = \begin{bmatrix} u_{11} & u_{12} & .. & u_{1k} & u_{1M} \\ u_{21} & u_{12} & .. & u_{2k} & u_{2M} \\ u_{31} & u_{12} & .. & u_{3k} & u_{3M} \\ : & : & .. & : & : \\ u_{N1} & u_{12} & .. & u_{Nk} & u_{NM} \end{bmatrix} \quad (4)$$

Therefore, a*ny* motion the structure may exhibit can be stated in terms of modal parameters:

$$\{x\} = [U]\{q\} \quad (5)$$

Where $\{q\}$ is a vector of **M** *modal coordinates*.

Experimental versus Analytical Modes

In experimental modal analysis, we start with a series of FRF measurements and work toward a model of the structure in terms of its modal parameters. In finite element analysis (FEA) and other analytic vibration studies, we seek exactly the same solution, but from a different starting point: a set of differential equations.

It is useful for the experimentalist to appreciate how mode shapes solve the *analytical* problem. This will also show the relationship between mode shapes and *residues*.

Finite element analysis is used to develop three **N** by **N** matrices, [M], [C] and [K], that describe the dynamics of a structure in terms of **N** degrees-of-freedom. A vector of **N** forces, {F} to be applied to the structure completes the model. These three matrices and force vector are arranged as the set of *second order differential equations*, shown below.

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\} \quad (6)$$

In general, these equations are *coupled* meaning that the off-diagonal terms in the matrices are non-zero. They are typically solved to yield a set of **N** modal vectors typified in form by (4).

Two *physical properties* are assumed in the matrices, [M], [C] and [K]. They are *symmetric* because the structure exhibits measurement *reciprocity* (this also implies that $H_{ij} = H_{ji}$). Second, [K] is assumed to be *positive definite* because the structure is *statically determinant* (does not require external forces to hold it in position).

Because of the *symmetric* and *positive definite* matrix properties, the solution vectors exhibit *generalized orthogonality* with respect to [M], that is:

$$\{u_k\}^T [M]\{u_k\} = m_k \quad (7a)$$

and:

$$\{u_k\}^T [M]\{u_m\} = \{u_m\}^T [M]\{u_k\} = 0 \quad (7b)$$

Where $\mathbf{m_k}$ is the *modal mass* of the $\mathbf{k}^{th}$ mode.

The importance of *generalized orthogonality* may be better appreciated by returning to the original problem statement (6) (with [**C**] assumed equal to [**0**]). All of the modal vectors may be grouped together as a transformation matrix in the manner of (5) and substituted in (6). Then *pre-multiplying* these equations by the *transpose* of the transformation results in:

$$\begin{aligned}&[U]^T[M][U]\{\ddot{q}\}+[U]^T[K][U]\{q\}=[U]^T\{F\}\\&=\left[\ddots m_k\right]\{\ddot{q}\}+\left[\ddots m_k\omega_k^2\right]\{q\}=[U]^T\{F\}=\{Q\}\end{aligned} \qquad (8)$$

Where $[U]^T$ is the matrix transpose of [U] and the vector, {Q}, is called a *generalized force*.

Note in (8) that the generalized orthogonality applies to both the mass and stiffness matrices, and that both resulting matrices are *diagonal*. The diagonal elements, m_k, are the *modal masses* (sometimes called *generalized masses*) of the system.

For lightly damped structures, even those exhibiting *complex modes*, it can also be shown that the damping matrix is diagonalized.

$$[U]^T[M][U]\cong\left[\ddots m_k\right] \qquad (9)$$

$$[U]^T[C][U]\cong\left[\ddots 2m_k\sigma_k\right] \qquad (10)$$

$$[U]^T[K][U]\cong\left[\ddots m_k\left(\sigma_k^2+\omega_k^2\right)\right] \qquad (11)$$

It is often convenient to scale modal vectors so that all of their associated modal masses are equal to **1** mass unit (kg, lb_m, etc.) Vectors scaled in this manner are said to be scaled to *unit modal mass* (UMM).

It is clear from equations (8) and (9) that modal mass is proportional to the square-root of the length (*magnitude*) of the mode shape vectors in both the *normal mode* and *complex mode* cases.

Another important property of modal vectors is that while their "shape" is unique, their values are not. Equations (9) to (11) show that modal mass and the mode shapes are related to one another. Both are arbitrary, but if a value is chosen for one, the other becomes fixed also.

RETURNING TO RESIDUES

The *residues* retain the physical units of an experimental modal analysis. Unlike modal mass, a residue is a *physical constant*. Its value is *independent* of any scaling applied to the mode shapes.

A residue is equal to two DOF elements of a *mode shape* vector divided by its *modal mass*. Specifically:

$$R_{ij}(k)=\frac{u_{ik}u_{jk}}{m_k}=\frac{u_{jk}u_{ik}}{m_k}=R_{ji}(k) \qquad (12)$$

Where u_{ik} = displacement of DOF, **i**, in the **k**th mode,
u_{jk} = displacement of DOF, **j**, in the **k**th mode and
m_k = modal mass of the **k**th mode.

It is clear then, that there is a *symmetric* matrix of residues, $[R_{ij}(k)]$, for *each* mode, **k**, measured from the structure. The symmetry of the residue matrices properly reflects *structural reciprocity*.

It is also clear that the residues reflect the mode shapes *redundantly*. Every *row* contains the complete mode shape multiplied the *response DOF* element of the vector. Likewise, every *column* contains the complete modal vector multiplied by the shape element for the *excitation* DOF.

WITH REFERENCE TO THE INITIAL QUESTION

For the *roving accelerometer test*, the FRFs measured, and consequently the residues and mode shape components obtained from them, are tabulated below.

FRFs		Residues		Mode Shape Components
$H_{1x,9z}$	→	$R_{1x,9z} = u_{1x}\,u_{9z}$	→	u_{1x}
$H_{1y,9z}$	→	$R_{1y,9z} = u_{1y}\,u_{9z}$	→	u_{1y}
$H_{1z,9z}$	→	$R_{1z,9z} = u_{1z}\,u_{9z}$	→	u_{1z}
$H_{2x,9z}$	→	$R_{2x,9z} = u_{2x}\,u_{9z}$	→	u_{2x}
$H_{2y,9z}$	→	$R_{2y,9z} = u_{2y}\,u_{9z}$	→	u_{2y}
$H_{2z,9z}$	→	$R_{2z,9z} = u_{2z}\,u_{9z}$	→	U_{2z}
•		•		•
•		•		•
•		•		•
$H_{9x,9z}$	→	$R_{9x,9z} = u_{9x}\,u_{9z}$	→	u_{9x}
$H_{9y,9z}$	→	$R_{9y,9z} = u_{9y}\,u_{9z}$	→	u_{9y}
$H_{9z,9z}$	→	$R_{9z,9z} = u_{9z}\,u_{9z}$	→	u_{9z}

When a *roving accelerometer test* is performed, the excitation DOF is fixed. That is, one component (the *reference* u_{jk} component) of each $R_{ij}(k)$ is the same. Each time we position the tri-axial accelerometer, we gain three new response u_{ik} components. Hence, this test measures a *single column* of the **M** residue matrices. In nine positionings, we measure 27 DOFs.

This column of FRFs will yield *one estimate* of each mode shape in **3** directions (**X**, **Y**, and **Z**) at each of the 9 points. The residues are also arranged in the *one column* of the residue matrix as it shown below.

Residue Matrix

1Z	2Z	3Z	•	•	•	9Z	
						$R_{1X,9Z}$	**1X**
						$R_{1Y,9Z}$	**1Y**
						$R_{1Z,9Z}$	**1Z**
						$R_{2X,9Z}$	**2X**
						$R_{2Y,9Z}$	**2Y**
						$R_{2Z,9Z}$	**2Z**
						•	•
						•	•
						•	•
						$R_{9X,9Z}$	**9X**
						$R_{9Y,9Z}$	**9Y**
						$R_{9Z,9Z}$	**9Z**

For the *roving impact test*, the FRFs measured, and consequently the residues and mode shape components obtained from them, are tabulated at right above.

When we perform a *roving impact test*, the response sensor remains at a fixed location, oriented in a fixed direction. Hence, for every impact measurement, the *reference* u_{ik} component of each R_{ijk} is the same. At each new strike point of the hammer, a different u_{jk} component is encountered. Hence this type of test measures one *row* of the residue matrix. When a tri-axial accelerometer is used, *three rows* are measured simultaneously as there are *three reference DOFs* at the fixed accelerometer site.

FRFs		Residues		Mode Shape Components
$H_{9x,1z}$	→	$R_{9x,1z} = u_{9x}\, u_{1z}$	→	u_{1z}
$H_{9y,1z}$	→	$R_{9y,1z} = u_{9y}\, u_{1z}$	→	u_{1z}
$H_{9z,1z}$	→	$R_{9z,1z} = u_{9z}\, u_{1z}$	→	u_{1z}
$H_{9x,2z}$	→	$R_{9x,2z} = u_{9x}\, u_{2z}$	→	u_{2z}
$H_{9y,2z}$	→	$R_{9y,2z} = u_{9y}\, u_{2z}$	→	u_{2z}
$H_{9z,2z}$	→	$R_{9z,2z} = u_{9z}\, u_{2z}$	→	u_{2z}
•		•		•
•		•		•
•		•		•
$H_{9x,9z}$	→	$R_{9x,9z} = u_{9x}\, u_{9z}$	→	u_{9z}
$H_{9y,9z}$	→	$R_{9y,9z} = u_{9y}\, u_{9z}$	→	u_{9z}
$H_{9z,9z}$	→	$R_{9z,9z} = u_{9z}\, u_{9z}$	→	u_{9z}

It is clear that the residues yield *three estimates* of each mode shape, each shape with 9 DOFs **1Z, 2Z, … 9Z**. The residues are arranged in *three rows* of the residue matrix, as shown below.

Residue Matrix

1Z	2Z	3Z	•	•	•	9Z	
							1X
							1Y
							1Z
							•
							•
							•
$R_{9X,1Z}$	$R_{9X,2Z}$	$R_{9X,3Z}$	•	•	•	$R_{9X,9Z}$	**9X**
$R_{9Y,1Z}$	$R_{9Y,2Z}$	$R_{9Y,3Z}$	•	•	•	$R_{9Y,9Z}$	**9Y**
$R_{9Z,1Z}$	$R_{9Z,2Z}$	$R_{9Z,3Z}$	•	•	•	$R_{9Z,9Z}$	**9Z**

Note, however, that the three rows contain only a single *driving-point* measurement (**9Z:9Z**). Any single row contains sufficient information to calculate the *modal frequency*, *modal damping* and *mode shape*. However, *modal mass* can only be calculated if the **9Z** row is included in the analysis.

With the *driving point* measurement, *UMM scaled* mode shapes can be determined. This is a matter of concern only if the results are to be used in a *Structural Dynamics Modification* (SDM).

REFERENCES

1. Richardson, M., *Modal Mass, Stiffness and Damping*, Vibrant Technology, Inc., January 2000.

1. Frazer, R. A., *Elementary Matrices*, The Macmillan Company, New York, NY, 1946.