



Averaging Linear and Auto Power Spectra

INTRODUCTION

In ME'scopeVES, Linear Spectra and Auto Power Spectra can be calculated from time domain signals. Both of these calculations include spectrum averaging, which includes choices of time domain window (Hanning, Flat Top, and Rectangular), number of averages, Peak Hold or Stable averaging, and Percent Overlap.

Spectrum averaging is normally done to reduce or remove the effects of random noise in the spectrum estimates. However, it will be shown in this Application Note that it averages noise out of a Linear Spectrum differently than from an Auto Power Spectrum (APS).

Spectrum averaging involves the computation of the Discrete Fourier Transform (DFT), or Fourier Spectrum. The DFT is computed with the FFT algorithm, and is a linear transformation. When the FFT is applied to a non-linear signal, the non-linearities will typically show up as high frequency noise in the Fourier Spectrum. This noise can be averaged out of the Linear and Auto Power Spectrum estimates.

FOURIER SPECTRUM

A Fourier Spectrum is computed by transforming a Time Response signal using the FFT. In ME'scopeVES, this is done by executing the **Transform | FFT** command in a Data Block window.

$$\mathbf{X}(\omega) = \text{FFT}(\mathbf{x}(t)) \quad (1)$$

When averaging multiple records of spectral data, we assume that the time domain signal and its Fourier Spectrum are really made up of the sum of two signals, the measured signal plus an additive random noise signal. The measured signal is assumed to be the same for all averages, while the noise portion is assumed to vary from one average to the next. Therefore, by summing together a series of positive and negative noise terms, averaging is used to reduce or eliminate the noise while preserving the measured (desired) signal.

Throughout this Application Note, we will break the *complex valued* Fourier Spectrum into 2 parts, the measured signal and the additive noise. The measured signal, $(A+jB)$, is assumed to remain constant for each average. The noise term, (C_i+jD_i) , is assumed to vary from one average to the

next. The i^{th} estimate of the Fourier Spectrum $\mathbf{X}_i(\omega)$ can then be written as,

$$\mathbf{X}_i(\omega) = A(\omega) + jB(\omega) + C_i(\omega) + jD_i(\omega) \quad (2)$$

LINEAR SPECTRUM

A Linear Spectrum is computed by averaging together one or more Fourier Spectra. *For one average, the Linear Spectrum is equal to the Fourier Spectrum.* For multiple averages, the Linear Spectrum corresponds to the average of the Fourier Spectra for each sampling window of time data.

For multiple averages, the Linear Spectrum is computed by averaging the real and imaginary terms of the Fourier Spectra. The Linear Spectrum can be written in terms of Fourier Spectrum components as,

$$\begin{aligned} \mathbf{X}_{LS}(\omega) &= \frac{1}{N} \sum_{i=1}^N \mathbf{X}_i(\omega) \\ &= \frac{1}{N} \sum_{i=1}^N (A(\omega) + jB(\omega) + C_i(\omega) + jD_i(\omega)) \\ &= A(\omega) + jB(\omega) + \frac{1}{N} \sum_{i=1}^N (C_i(\omega) + jD_i(\omega)) \end{aligned} \quad (3)$$

This is the process used in ME'scopeVES for computing Linear Spectra with Stable averaging.

AUTO POWER SPECTRUM (APS)

The APS is computed by multiplying the Fourier Spectrum by its complex conjugate. For multiple averages, the resulting APS is the average of multiple APS estimates, one for each sampling window of time domain data.

The APS calculation produces only a magnitude value, the average of multiple estimates of the APS is merely the average of multiple magnitudes, one from each time sampling window.

Equation (4) is used in ME'scopeVES for computing APS, with Stable averaging. Note that in the last term of equation (4), the average magnitude of the noise is added to the APS. Since this (noise power) term is always positive, there is no way to completely remove the effects of noise from the averaged APS estimate.

$$\begin{aligned}
\mathbf{X}_{\text{APS}}(\omega) &= \frac{1}{N} \sum_{i=1}^N \mathbf{X}_i(\omega) \mathbf{X}_i^*(\omega) \\
&= \frac{1}{N} \sum_{i=1}^N (\mathbf{A}(\omega) + \mathbf{jB}(\omega) + \mathbf{C}_i(\omega) + \mathbf{jD}_i(\omega)) (\mathbf{A}(\omega) - \mathbf{jB}(\omega) + \mathbf{C}_i(\omega) - \mathbf{jD}_i(\omega)) \\
&= \mathbf{A}^2(\omega) + \mathbf{B}^2(\omega) + \frac{1}{N} \sum_{i=1}^N (2\mathbf{A}(\omega)\mathbf{C}_i(\omega) + 2\mathbf{B}(\omega)\mathbf{D}_i(\omega) + \mathbf{C}_i^2(\omega) + \mathbf{D}_i^2(\omega)) \\
&= \mathbf{A}^2(\omega) + \mathbf{B}^2(\omega) + \frac{2\mathbf{A}(\omega)}{N} \sum_{i=1}^N \mathbf{C}_i(\omega) + \frac{2\mathbf{B}(\omega)}{N} \sum_{i=1}^N \mathbf{D}_i(\omega) + \frac{1}{N} \sum_{i=1}^N (\mathbf{C}_i^2(\omega) + \mathbf{D}_i^2(\omega))
\end{aligned} \tag{4}$$

$$\begin{aligned}
\mathbf{X}_{\text{APS}}(\omega) &= \mathbf{X}_{\text{LS}}(\omega) \mathbf{X}_{\text{LS}}^*(\omega) \\
&= \left(\mathbf{A}(\omega) + \mathbf{jB}(\omega) + \frac{1}{N} \sum_{i=1}^N (\mathbf{C}_i(\omega) + \mathbf{jD}_i(\omega)) \right) \left(\mathbf{A}(\omega) - \mathbf{jB}(\omega) + \frac{1}{N} \sum_{i=1}^N (\mathbf{C}_i(\omega) - \mathbf{jD}_i(\omega)) \right) \\
&= \mathbf{A}^2(\omega) + \mathbf{B}^2(\omega) + \frac{2\mathbf{A}(\omega)}{N} \sum_{i=1}^N (\mathbf{C}_i(\omega)) + \frac{2\mathbf{B}(\omega)}{N} \sum_{i=1}^N (\mathbf{D}_i(\omega)) + \left(\frac{1}{N} \sum_{i=1}^N \mathbf{C}_i(\omega) \right)^2 + \left(\frac{1}{N} \sum_{i=1}^N \mathbf{D}_i(\omega) \right)^2
\end{aligned} \tag{5}$$

Computing the APS from the Linear Spectrum

The APS can also be computed from the Linear Spectrum, simply by multiplying the Linear Spectrum by its complex conjugate. Using Equation (3) to compute the Linear Spectrum gives the result shown in Equation (5).

When the APS is computed using the Linear Spectrum, the results are the same as equation (4), except for the term that averages the noise with itself. Using the traditional method (Equation 4) yields a term that is the *average magnitude of the noise*. When the Linear Spectrum is used, the average magnitude of the noise term is replaced by the square of the average real noise value plus the square of the average imaginary noise value.

REMOVAL OF NOISE

Since noise is assumed to be random, we expect some real component terms (C_i) to be positive and some to be negative. With sufficient averages, we expect *the average of the C_i values to tend towards zero*. The same is true for the imaginary components (D_i). (This is a characteristic of Gaussian random noise.) The *average magnitude* of the noise does not average to zero, however.

$$\begin{aligned}
\lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{i=1}^N \mathbf{C}_i(\omega) \right) &= 0, \quad \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{i=1}^N \mathbf{C}_i^2(\omega) \right) \neq 0 \\
\lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{i=1}^N \mathbf{D}_i(\omega) \right) &= 0, \quad \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{i=1}^N \mathbf{D}_i^2(\omega) \right) \neq 0
\end{aligned} \tag{6}$$

With sufficient averages, some of the noise terms can be removed from all of the average estimates.

Linear Spectrum with Noise Removed

Removing the zero terms (6) from Equation (3) gives,

$$\mathbf{X}_{\text{LS}}(\omega) = \mathbf{A}(\omega) + \mathbf{jB}(\omega) \tag{7}$$

This shows that averaging Linear Spectrum estimates *completely removes noise from the final result*, if enough averages are used.

APS with Noise Removed

Removing the zero terms (6) from Equation (4) gives,

$$\mathbf{X}_{\text{APS}}(\omega) = \mathbf{A}^2(\omega) + \mathbf{B}^2(\omega) + \frac{1}{N} \sum_{i=1}^N (\mathbf{C}_i^2(\omega) + \mathbf{D}_i^2(\omega)) \tag{8}$$

Equation (8) shows that the *noise power sum stays in* the averaged APS estimate.

Auto Spectrum From Linear Spectrum

Removing the zero terms (6) from Equation (5) gives,

$$\mathbf{X}_{\text{APS}}(\omega) = \mathbf{A}^2(\omega) + \mathbf{B}^2(\omega) \tag{9}$$

Equation (9) shows that using the Linear Spectrum to compute the APS yields a noise free estimate, if sufficient averages are taken.

CONCLUSION

The APS can be computed directly from Fourier Spectra, or it can be computed using a Linear Spectrum. The advantage of using the Linear Spectrum is improved noise removal.