Vibrant Technology, Inc.

ME'scopeVES Application Note #16

Integration & Differentiation of FRFs

INTRODUCTION

Integration & differentiation of time or frequency domain waveforms can be performed in any Data Block window in ME'scopeVES. In addition, integration & differentiation of residue mode shapes can be performed in the Shape Table window containing the shapes.

In this note, the relationships between integration & differentiation of FRFs, and integration & differentiation of modal parameters will be developed.

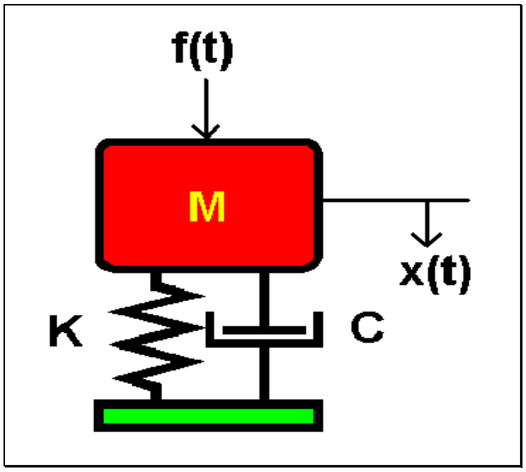


Figure 1. Mass-Spring-Damper.

The modal properties of real world structures are analyzed using a multi-degree-of-freedom (MDOF) dynamic model. In this note, the mass-spring-damper model shown in Figure 1 will be used. This is a single degree-of-freedom (SDOF) model. Nevertheless, the dynamics of MDOF structures are better understood by analyzing the dynamics of this SDOF structure.

The dynamic behavior of the mass-spring-damper structure in Figure 1 is represented by a single (scalar) equation, Equation 1. (An MDOF structure is represented by multiple equations like Equation 1, which are written in matrix form.)

Modes are defined for structures, the dynamics of which can be represented by linear ordinary differential equations like Equation 1. Because of the superposition property of linear systems, the dynamics of an MDOF structure can be written as a ***summation of contributions due to each of its modes***. Each mode can be thought of as representing the dynamics of a single mass-spring-damper system.

BACKGROUND MATH

The time domain equation of motion for the mass-spring-damper is represented by Newton's Second Law,

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t) \qquad (1)$$

where:

$M =$ **mass value**

$C =$ **damping coefficient**

$K =$ **spring stiffness**

$\ddot{x}(t) =$ **acceleration**

$\dot{x}(t) =$ **velocity**

$x(t) =$ **displacement**

$f(t) =$ **excitation force**

Laplace Transforms

By taking Laplace transforms of the terms in Equation 1 and setting initial conditions to zero, an equivalent frequency domain equation of motion results,

$$[M s^2 + C s + K]\ X(s) = F(s) \qquad (2)$$

where: $X(s)$ = Laplace transform of the displacement

$F(s)$ = Laplace transform of the force

$s = \sigma + j\omega =$ complex Laplace variable

Transfer Function

Equation 2 can be rewritten by simply dividing both sides by the coefficients of the left-hand side.

$$X(s) = \left(\frac{1}{M s^2 + C s + K} \right) F(s) \qquad (3)$$

The new coefficient on the right hand side of Equation 3 is called the Transfer Function**,**

$$H(s) = \frac{X(s)}{F(s)} = \left(\frac{1}{M s^2 + C s + K} \right) \qquad (4)$$

The Transfer Function is complex valued, and therefore has two parts; ***real & imaginary*** or equivalently ***magnitude & phase***. The two parts of the Transfer Function can be plot-

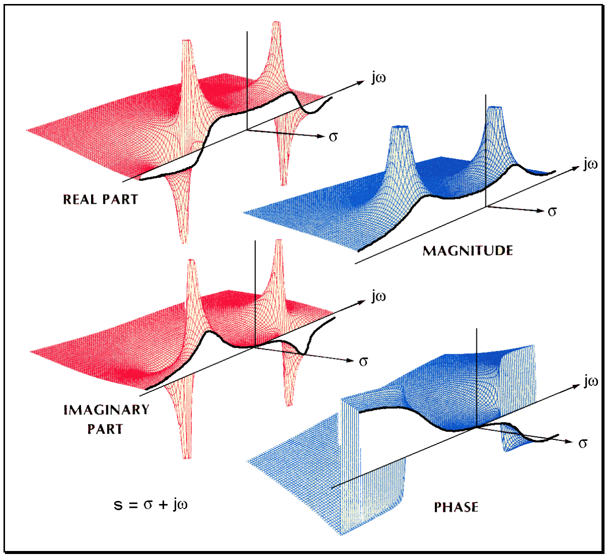


Figure 2. Transfer Function on the s-Plane.

ted on the complex Laplace plane (or **s**-plane), as shown in Figure 2.

Poles of the Transfer Function

Notice that the magnitude of the Transfer Function has two peaks in it. These are points where the value of the Transfer Function ***goes to infinity***. (The real and imaginary parts also show the same two peaks.)

By inspection of Equation 4, it is clear that the Transfer Function goes to infinity for values on the **s**-Plane where its ***denominator is zero***. It is also clear that as **s** goes to infinity, the Transfer Function will go to zero.

The denominator is a second order polynomial in the **s** variable, called the ***characteristic polynomial***. Since it is a second order, it has two roots (values of **s**) for which it will be zero. These two roots of the denominator are called the ***poles*** of the Transfer Function. Furthermore, the poles are complex conjugates of one another. The poles therefore, are the locations in the **s**-plane where the Transfer Function has a value of infinity. The poles are also called **eigenvalues**.

$$\mathbf{p_0 = -\sigma_0 + j\omega_0\ ,\quad p_0{}^* = -\sigma_0 - j\omega_0} \qquad (5)$$

s-Plane Nomenclature

The real axis in the **s**-Plane is called the ***damping axis***, and the imaginary axis is called the ***frequency axis***. The locations of the poles in the **s**-Plane have also been given some other commonly used names, as shown in Figure 3.

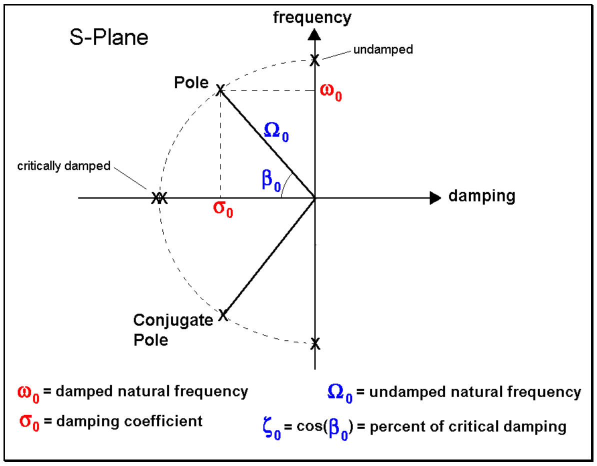


Figure 3. s-Plane Nomenclature.

Modal Parameters

The coordinates of the poles in the **s**-Plane are also modal parameters. Rewriting Equation 4 in terms of its pole locations, or modal parameters,

$$\mathbf{H(s) = \left(\frac{1/M}{s^2 + 2\sigma_0 s + \Omega_0{}^2}\right)} \qquad (6)$$

where: $$\mathbf{\sigma_0 = \frac{C}{2\,M}\ ,\quad \Omega_0{}^2 = \frac{K}{M}} \qquad (7)$$

$$\mathbf{\sigma_0 = \text{modal damping coefficient}}$$

$$\mathbf{\Omega_0 = \text{undamped modal frequency}}$$

$$\mathbf{\Omega_0{}^2 = \sigma_0{}^2 + \omega_0{}^2}$$

$$\mathbf{\omega_0 = \text{damped modal frequency}}$$

And, the ***percent of critical damping*** (ζ_0) is written as,

$$\zeta_0 = \frac{\sigma_0}{\Omega_0} = \frac{C}{2\sqrt{MK}} \qquad (8)$$

Frequency Response Function (FRF)

Notice that in Figure 2 ***the Transfer Function has only been plotted for half of the s-Plane.*** That is, it has only been plotted for negative values of σ (the real part of **s**). This was done so that the values of the Transfer Function along the **jω**-axis (the imaginary part of **s**) are clearly seen.

Definition: The **Frequency Response Function** (**FRF**) is the values of the Transfer Function along the **jω**-axis.

The FRF values are pointed out in Figure 4. Since the FRF is only defined along the **jω**-axis, **s** can be expressed in terms of **jω**,

$$\mathbf{FRF} = \mathbf{H(j\omega)} = \mathbf{H(s)}\Big|_{s=j\omega}$$

$$= \frac{\mathbf{X(s)}}{\mathbf{F(s)}}\bigg|_{s=j\omega} = \frac{\mathbf{X(j\omega)}}{\mathbf{F(j\omega)}} \tag{9}$$

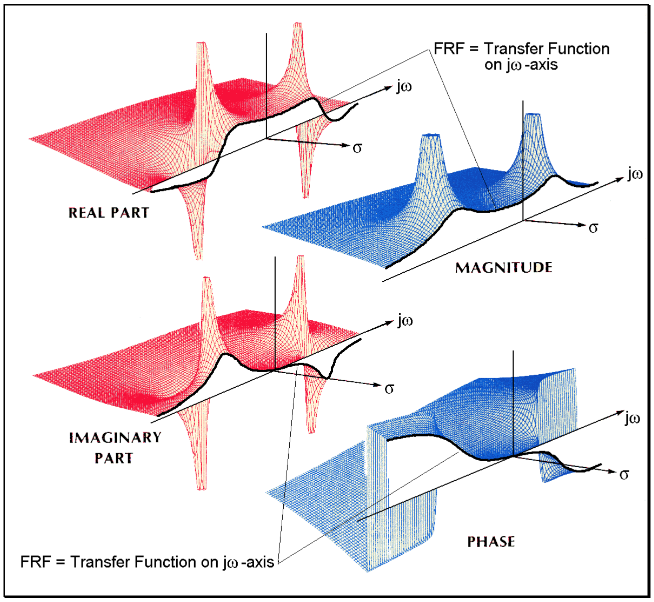


Figure 4. FRF Plotted on the jω-axis.

FRF in Partial Fraction Form

The FRF for an SDOF can now be written by simply replacing the **s**-Variable in Equation 6 with $\mathbf{j\omega}$,

$$\mathbf{H(j\omega)} = \frac{\mathbf{(1/M)}}{\mathbf{(j\omega)^2 + 2\sigma_0(j\omega) + \Omega_0{}^2}} \tag{10}$$

Furthermore, using the poles of the Transfer Function, a partial fraction expansion can be performed on Equation 10 yielding,

$$\mathbf{H(j\omega)} = \frac{\mathbf{1}}{\mathbf{2j}}\left[\frac{\mathbf{R_0}}{\mathbf{j\omega - p_0}} - \frac{\mathbf{R_0}}{\mathbf{j\omega - p_0\,*}}\right] \tag{11}$$

where: $\mathbf{R_0} = \mathbf{1/\omega_0 M}$ (12)

$\mathbf{R_0}$ is called the modal ***residue***. It is simply the amplitude (or strength) of the numerator of each resonance term in Equation 11. Comparing Equation 11 with the FRF in Figure 4, it is clear that the FRF of an SDOF is the ***summation of two resonance curves***, each one forming a peak due to one of the two pole locations,

$$\mathbf{p_0 = -\sigma_0 + j\omega_0} \ \& \ \mathbf{p_0{*} = -\sigma_0 - j\omega_0}.$$

Equation 11 also says that the FRF for an SDOF is ***fully represented by two poles and two residues***. Furthermore, since the residues are equal and the poles are complex conjugates of one another, the complete dynamics of an SDOF system are ***fully represented by a modal frequency*** $\mathbf{(j\omega_0)}$, ***modal damping*** $\mathbf{(\sigma_0)}$, and ***modal residue*** $\mathbf{(R_0)}$.

Mode Shapes

One final step is to represent the FRF in terms of mode shapes instead of residues,

$$\mathbf{H(j\omega)} = \frac{\mathbf{1}}{\mathbf{2j}}\left[\frac{\mathbf{\{u_0\}^2}}{\mathbf{j\omega - p_0}} - \frac{\mathbf{\{u_0\}^2}}{\mathbf{j\omega - p_0\,*}}\right] \tag{13}$$

where:

$$\mathbf{\{u_0\}} = \begin{Bmatrix} \dfrac{\mathbf{1}}{\sqrt{\mathbf{A\omega_0 M}}} \\ \mathbf{0} \end{Bmatrix}, \ \mathbf{A = scaling\ constant} \tag{14}$$

Notice that the mode shape $\mathbf{\{u_0\}}$ is a vector, and that its second component is zero. This corresponds to the ***ground***, where there is no motion. Notice also that the mode shape contains a scaling constant (**A**). This is because a ***mode shape doesn't have unique values. Only its shape (one component relative to another) is unique***. A mode shape is also called an ***eigenvector***.

Equation 13 states that the dynamics of an SDOF are fully represented by a ***pair of eigenvalues (poles)*** and a ***pair of eigenvectors (mode shapes)***. Because of symmetry, Equation 13 also says that the dynamics is fully represented by a ***modal frequency*** $\mathbf{(j\omega_0)}$, ***modal damping*** $\mathbf{(\sigma_0)}$, and a ***mode shape*** $\mathbf{\{u_0\}}$.

Residue Units

From Equation 11, it is clear that,

Residue units = (**FRF units**) x (**Radians/Second**)

This is because the units of the FRF denominator are (radians/second), or $\mathbf{2\pi}$ Hz.

Impulse Response Function (IRF)

The Impulse Response Function is the Inverse FFT of the FRF. It too can be written in terms of modal parameters and provides the best source of meaning for the modal parameters. Using Equation 11, which expresses the FRF in terms of modal parameters, the IRF is written,

$$\mathbf{h(t) = FFT^{-1}\left(\frac{1}{2j}\left[\frac{R_0}{j\omega - p_0} - \frac{R_0}{j\omega - p_0\,*}\right]\right)} \qquad (15)$$

or

$$\mathbf{h(t) = \frac{T}{2j}\left[R_0 e^{p_0 t} - R_0 e^{p_0 * t}\right]} \qquad (16)$$

or

$$\mathbf{h(t) = T\,|R_0|\,e^{-\sigma_0 t}\left(\sin(\omega_0 t + \alpha_0)\right)} \qquad (17)$$

where: $\mathbf{\alpha_0 = \text{phase angle of } R_0}$

Equation 17 shows clearly the role that each modal parameter plays in the IRF. $\mathbf{\omega_0}$ multiplies the time variable ($\mathbf{t}$) in the sinusoidal function $\mathbf{(\sin(\omega_0 t + \alpha_0))}$. It defines the frequency of oscillation, hence $\mathbf{\omega_0}$ is called ***modal frequency***.

$\mathbf{\sigma_0}$ is the coefficient in the exponential term $\mathbf{(e^{-\sigma_0 t})}$ that defines the envelope of decay for the IRF. Since the decay is caused by a combination of damping mechanisms within or about a structure, $\mathbf{\sigma_0}$ is called the ***modal damping***, or damping coefficient.

IRF Units

From Equation 17, it is clear that the IRF is the product of two functions, an ***exponential*** function $\mathbf{(T\,|R_0|\,e^{-\sigma_0 t})}$ and a ***sinusoidal*** function $\mathbf{(\sin(\omega_0 t + \alpha))}$. The exponential decay $\mathbf{(e^{-\sigma_0 t})}$ and sinusoidal function are dimensionless and only **T** and $\mathbf{R_0}$ have units. Therefore,

Impulse Response units = (**Seconds**) x (**Residue units**)

or,

Impulse Response units = **FRF units**

Differentiating IRFs

Differentiating the **(displacement/force) IRF** in Equation 16 with respect to time yields Equation 18.

$$\mathbf{\begin{aligned}\frac{dh(t)}{dt} &= \frac{d}{dt}\left(\frac{T}{2j}\left[R_0 e^{p_0 t} - R_0 e^{p_0^* t}\right]\right)\\ &= \frac{T}{2j}\left[R_0\frac{d\left(e^{p_0 t}\right)}{dt} - R_0\frac{d\left(e^{p_0^* t}\right)}{dt}\right]\\ &= \frac{T}{2j}\left[R_0 p_0 e^{p_0 t} - R_0 p_0^* e^{p_0^* t}\right]\end{aligned}} \qquad (18)$$

It is clear from Equation 18 that ***differentiating the IRF is equivalent to multiplying its modal residues by their respective poles***. In other words,

$$\mathbf{\begin{aligned}R_{Velocity} &= R_{Displacement}\;p\\ R^*_{Velocity} &= R^*_{Displacement}\;p^*\end{aligned}} \qquad (19)$$

NOTE: For this SDOF system both poles have real residues $\mathbf{R_0}$, but for MDOF systems in general, each mode has a complex conjugate pair of residues.

Differentiating the **(velocity/force) IRF** in Equation 18 gives,

$$\mathbf{R_{Acceleration} = R_{Velocity}\,p = R_{Displacement}\;p^2} \qquad (20)$$

and similarly for the conjugate residue. From Equation 20, expressions relating (displacement/force-sec), (velocity/force-sec), and (acceleration/force-sec) residues can be written,

$$\mathbf{\begin{aligned}R_{Acceleration} &= R_{Velocity}\,p = R_{Displacement}\;p^2\\ R_{Velocity} &= R_{Displacement}\;p = \frac{R_{Acceleration}}{p}\\ R_{Displacement} &= \frac{R_{Velocity}}{p} = \frac{R_{Acceleration}}{p^2}\end{aligned}} \qquad (21)$$

and similarly for the conjugate residue.

In general, we can conclude that IRFs (and also FRFs) can be differentiated & integrated by applying Equations 21 to their modal residues.

Differentiating FRFs

In an ME'scopeVES Data Block window, FRFs (or any frequency domain function) are differentiated by multiplying them by the frequency variable $j\omega$. So one might ask. ***"Is multiplying an FRF by*** $j\omega$ ***equivalent to multiplying its modal residues by their pole locations?"***

A (**displacement/force**) Transfer Function can be differentiated by using the following differentiation formula for Laplace Transforms,

$$
\begin{aligned}
L\{h'(t)\} &= sH(s) - h(0^+) \\
&= \frac{R_0}{2j}\left[\frac{s(p_0 - p_0^*)}{(s-p_0)(s-p_0^*)}\right] - \frac{R_0}{2j}\left[e^0 - e^0\right] \\
&= \frac{R_0}{2j}\left[\frac{s(p_0 - p_0^*) + p_0 p_0^* - p_0 p_0^*}{(s-p_0)(s-p_0^*)}\right] \qquad (22) \\
&\quad - \frac{R_0}{2j}\left[1-1\right] \\
&= \frac{1}{2j}\left[\frac{R_0 p_0}{s-p_0} - \frac{R_0 p_0^*}{s-p_0^*}\right]
\end{aligned}
$$

The derivative of the FRF is obtained by evaluating Equation 22 along the frequency axis, $s = j\omega$. Equation 22 shows the same relationship between the (velocity/force-sec) residues and (displacement/force-sec) residues as previously obtained by differentiating the IRF, namely Equation 19.

Now if a (displacement/force) FRF is multiplied by $j\omega$, the FRF for the SDOF system in Equation 10 becomes,

$$H_{Velocity}(j\omega) = \frac{(j\omega / M)}{(j\omega)^2 + 2\sigma_0\, j\omega + \Omega_0^{\,2}} \qquad (23)$$

Performing a partial fraction expansion of this FRF gives the expected result,

$$R_{Velocity} = R_{Displacement}\; p_0 \qquad (24)$$

and similarly for the conjugate pole. So, ***multiplying a (displacement/force) FRF by*** $j\omega$ ***is the same as multiplying its residues by their respective poles***.

Now, consider double differentiation to obtain an (acceleration/force) FRF from a (displacement/force) FRF. Applying the formula for double differentiation of a Laplace Transform,

$$
\begin{aligned}
L\{h''(t)\} &= s^2 H(s) - sh(0^+) - h'(0^+) \\
&= \frac{R_0}{2j}\left[\frac{s^2(p_0 - p_0^*)}{(s-p_0)(s-p_0^*)}\right] - 0 - \frac{R_0}{2j}\left[p_0 - p_0^*\right] \\
&= \frac{R_0}{2j}\left[\frac{p_0 p_0 (s - p_0^*)}{(s-p_0)(s-p_0^*)} - \frac{p_0^* p_0^* (s - p_0)}{(s-p_0)(s-p_0^*)}\right] \qquad (24) \\
&= \frac{1}{2j}\left[\frac{R_0 p_0^{\,2}}{s-p_0} - \frac{R_0 p_0^{*\,2}}{s-p_0^*}\right]
\end{aligned}
$$

The second derivative of the FRF is obtained by evaluating Equation 24 along the frequency axis, $s = j\omega$. In addition, applying the Inverse FFT to Equation 24 gives the same result as previously obtained by double differentiating the IRF.

Now if a (displacement/force) FRF in Equation 10 is multiplied by $(j\omega)^2$, the result is,

$$H_{Acceleration}(j\omega) = \frac{\left((j\omega)^2 / M\right)}{(j\omega)^2 + 2\sigma_0\, j\omega + \Omega_0^{\,2}} \qquad (25)$$

Performing a partial fraction expansion of this FRF gives a ***different result*** than Equation 24,

$$H(j\omega) = \frac{1}{M} + \frac{1}{2j}\left[\frac{R_{Accel}}{j\omega - p_0} - \frac{R_{Accel}}{j\omega - p_0{}^*}\right] \qquad (26)$$

where,

$$R_{Acceleration} = R_{Displacement}\; p_0^{\,2} \qquad (27)$$

Since the numerator and denominator of Equation 25 are of the same order, the partial fraction expansion yields an extra term, (**1/M**).

What happened to this extra term in Equation 24? It was negated by the initial condition of the (velocity/force) IRF,

$$
\begin{aligned}
h'(0^+) &= \frac{R_0}{2j}\left[p_0 e^0 - p_0^* e^0\right] \\
&= \frac{R_0}{2j}\left[-\sigma_0 + j\omega_0 - (-\sigma_0 - j\omega_0)\right] \qquad (28) \\
&= \frac{1}{2jM\omega_0}\left[2j\omega_0\right] \\
&= \frac{1}{M}
\end{aligned}
$$

So, ***multiplying a (displacement/force) FRF by*** $(j\omega)^2$ ***is not the same as multiplying its residues by the square of their respective poles***.

Since the initial condition of the (velocity/force) IRF is not zero, it also follows that, ***multiplying a (velocity/force) FRF by*** $j\omega$ ***is not the same as multiplying its residues by their respective poles***.

Which one is correct? Multiplying & dividing modal residues by powers of their respective poles always gives the correct result.

Multiplying a (displacement/force) FRF by $j\omega$ to obtain a (velocity/force) FRF is correct. Likewise, a (displacement/force) FRF can be obtained by dividing a (velocity/force) FRF by $j\omega$. This is because the initial condition for a (displacement/force) IRF $h(0^+)$ is zero.

However, an (acceleration/force) FRF cannot be obtained by multiplying either a (displacement/force) FRF by $(j\omega)^2$ or a (velocity/force) FRF by $j\omega$. Likewise, an (acceleration/force) FRF cannot be divided by $j\omega$ to obtain a (velocity/force) FRF, or by $(j\omega)^2$ to obtain a (displacement/force) FRF. This is because the initial condition for a (velocity/force) IRF $h'(0^+)$ is non-zero.

A NUMERICAL EXAMPLE

The dynamic model for a mass-spring-damper system that is built in **Application Note #7 Modal Analysis of a Mass-Spring-Damper System** will also be used here.

The ***Visual* Modal, Modal Pro** or ***Visual* SDM** option is required to carry out all steps of this exercise. If you have the ***Visual* SDM** option, you can add mass, damping & stiffness elements directly to the SDOF model and generate its modal parameters. Otherwise, you can create a new Shape Table file and enter the modal parameters into it.

To create a new Shape Table,

- Execute **File | New | Shape Table** from the ME'scopeVES window. A dialog box will open. Type **SDOF mode** and click on **OK**. Another dialog box will open.
- Enter **number of shapes =1** and **number of DOFs =1**, and click on **OK**.
- Enter the following frequency & damping into the Shape Header spreadsheet,

$$\omega_0 = 9.9875 \text{ Hz}$$

$$\sigma_0 = 0.5\text{Hz} = 5\%$$

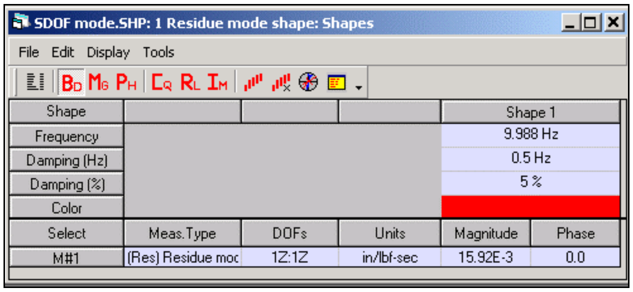
- Enter the following into the Shapes spreadsheet.

Measurement Type = Residue mode shape

DOFs = 1Z:1Z

Units = in/lbf-sec

$$R_0 = 0.015915$$



Shape				Shape 1
Frequency				9.988 Hz
Damping (Hz)				0.5 Hz
Damping (%)				5 %
Color				🟥

Select	Meas.Type	DOFs	Units	Magnitude	Phase
M#1	(Res) Residue moc	1Z:1Z	in/lbf-sec	15.92E-3	0.0

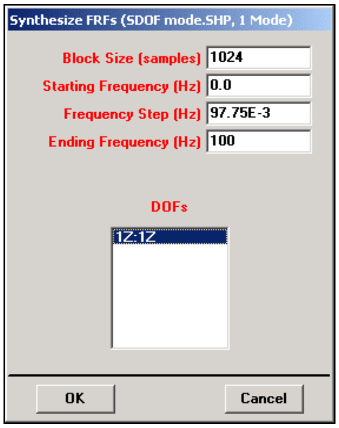
Figure 5. Shape Table with SDOF Mode.

Synthesizing FRFs Using Modal Parameters

The FRF, like a Transfer Function, defines the dynamic characteristics between two DOFs of a structure. MDOF systems have many DOF pairs for which FRFs can be derived or measured. The SDOF system in Figure 1 has only one meaningful DOF. Therefore, using modal parameters we can synthesize an FRF between the DOF (1Z) and itself. ***Any FRF between a DOF and itself is called a driving point FRF***.

Displacement/Force FRF

- Execute **Tools | Synthesize FRFs** in the Shape Table window. The FRF Synthesis dialog box will open.
- Edit the ending frequency to **100** Hz, select the driving point DOFs (**1Z:1Z**), and click on **OK**.



Block Size (samples) 1024
Starting Frequency (Hz) 0.0
Frequency Step (Hz) 97.75E-3
Ending Frequency (Hz) 100
1Z:1Z

Figure 6. Synthesize FRF Dialog Box.

- Click on **OK** again to accept the default Data Block name
- Next, the new Data Block window will open, showing the FRF.
- Execute **Display | Bode | Upper/Lower** in the new Data Block window to display the synthesized FRF.

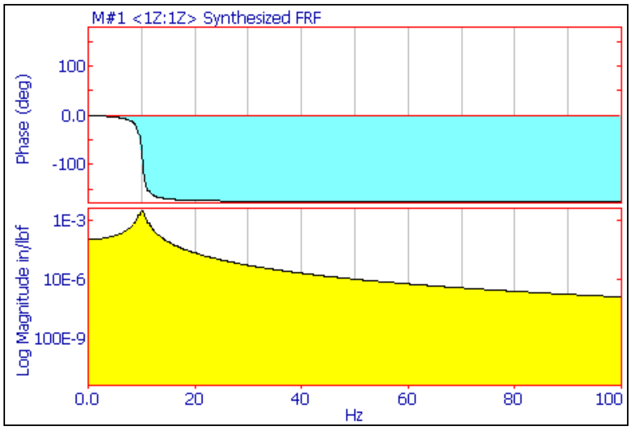
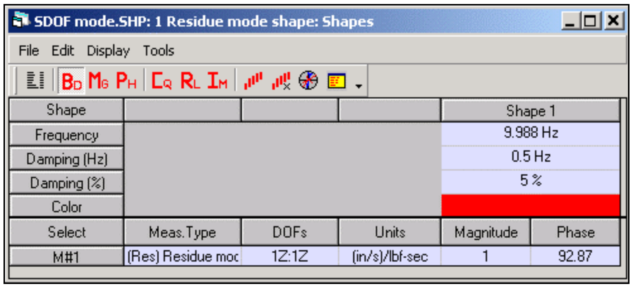


Figure 7. (Displacement/Force) FRF.

Velocity/Force FRF

A (velocity/force) FRF can be obtained in two ways; differentiating the residue mode shape and synthesizing the FRF, or multiplying the (displacement/force) FRF by $j\omega$.

- Execute **Tools | Differentiate** to calculate the (velocity/force-sec) residue in the Shape Table window.



Shape				Shape 1	
Frequency				9.988 Hz	
Damping (Hz)				0.5 Hz	
Damping (%)				5 %	
Color					
Select	Meas.Type	DOFs	Units	Magnitude	Phase
M#1	(Res) Residue moc	1Z:1Z	(in/s)/lbf-sec	1	92.87

Figure 8. (Velocity/Force-sec) Residue Mode Shape.

- Execute **Tools | Synthesize FRFs** and enter the same parameters as those in Figure 6.

Figure 9 shows the resulting (velocity/force) FRF.

Acceleration/Force FRF

- Execute **Tools | Differentiate** again from the Shape Table to calculate the (acceleration/force-sec) residue in Figure 10.
- Execute **Tools | Synthesize FRFs.** Figure 11 shows the resulting (acceleration/force) FRF.

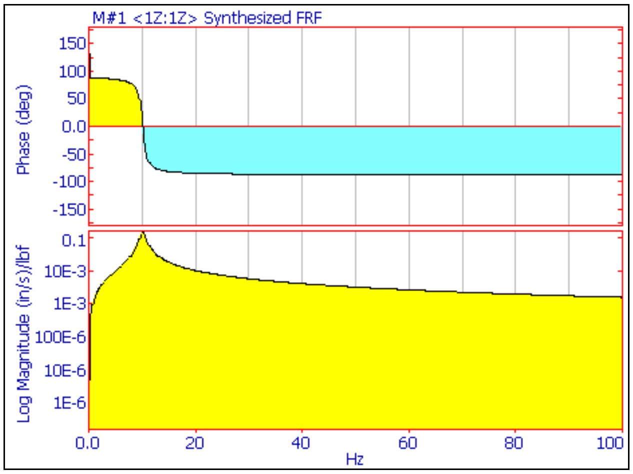
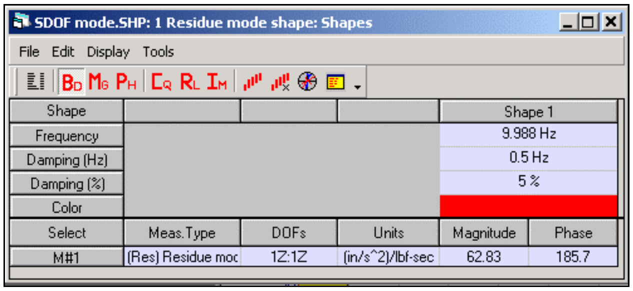


Figure 9. (Velocity/Force) FRF.



Shape				Shape 1	
Frequency				9.988 Hz	
Damping (Hz)				0.5 Hz	
Damping (%)				5 %	
Color					
Select	Meas.Type	DOFs	Units	Magnitude	Phase
M#1	(Res) Residue moc	1Z:1Z	(in/s^2)/lbf-sec	62.83	185.7

Figure 10. (Acceleration/Force-sec) Residue Mode Shape.

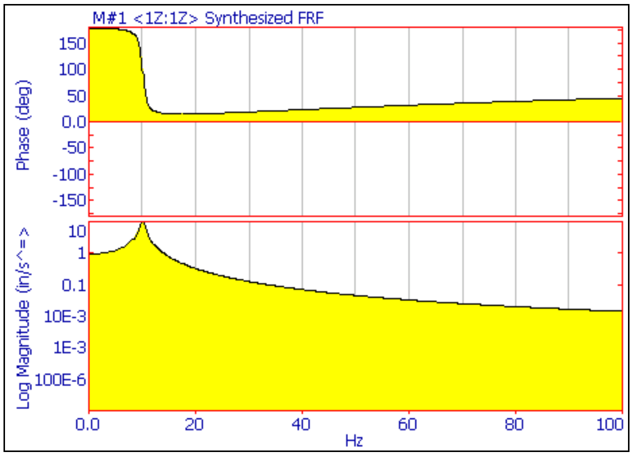


Figure 11. (Acceleration/Force) FRF.

Multiplying By jω

Now, the (displacement/force) FRF will be multiplied by $j\omega$ to compare it with the FRF in Figure 9.

- In the Data Block with the (displacement/force) FRF in it, execute **Tools | Math | Differentiate**.
- Execute **Edit | Paste Traces.** A dialog box will open. Select the Data Block with the (velocity/force) FRF, and click on **OK**.
- Execute **Format | Overlay** to compare the two FRFs. They should be nearly identical.

Next, the (velocity/force) FRF will be multiplied by $j\omega$ to compare it with Figure 11.

- In the Data Block with the (velocity/force) FRF in it, execute **Tools | Math | Differentiate**.
- Execute **Edit | Paste Traces.** A dialog box will open. Select the Data Block with the (acceleration/force) FRF, and click on **OK**.
- Execute **Format | Overlay** to compare the two FRFs.

Figure 12 shows the overlaid FRFs. The additional mass line (at "1") is clearly visible in the FRF obtained by multiplying the (velocity/force) FRF by $j\omega$.

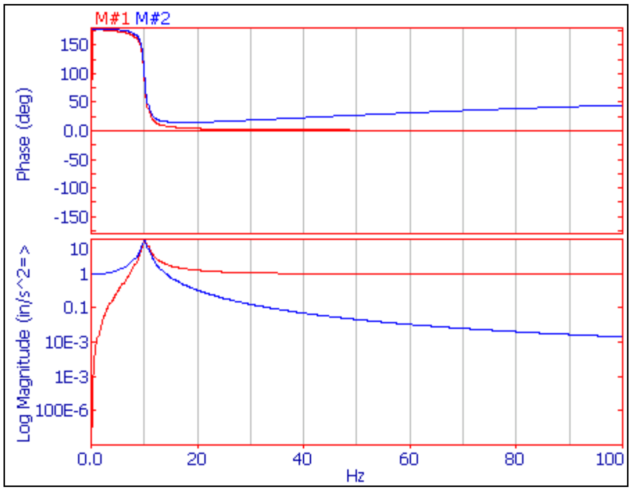


Figure 12. Overlaid (Acceleration/Force) FRFs.

Dividing By jω

Now, the (acceleration/force) FRF will be divided by $j\omega$ to compare it with the (velocity/force) FRF in Figure 9.

- In the Data Block with the (acceleration/force) FRF in it, execute **Tools | Math | Integrate**.
- Execute **Edit | Paste Traces.** A dialog box will open. Select the Data Block with the (velocity/force) FRF, and click on **OK**.
- Execute **Format | Overlay** to compare the two FRFs.

Figure 13 shows the overlaid FRFs. The additional mass line (at "1") is clearly visible in the FRF obtained by multiplying the (velocity/force) FRF by $j\omega$.

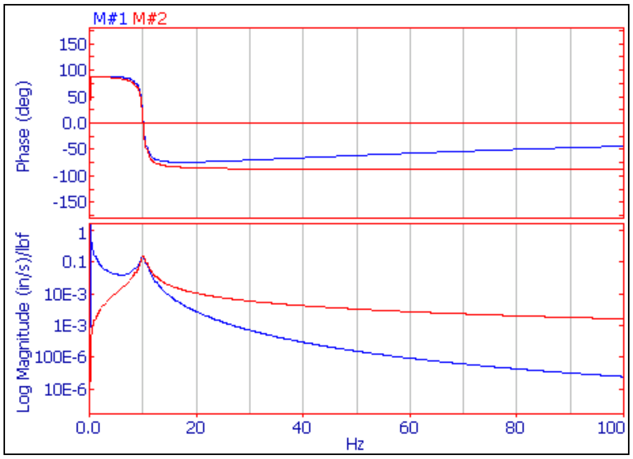


Figure 13. Overlaid (Velocity/Force) FRFs.

Conclusions

The following rules can be derived from this exercise,

1. Differentiating an FRF is the same as multiplying its residues by their respective poles.
2. Integrating an FRF is the same as dividing its residues by their respective poles.
3. Multiplying a (displacement/force) FRF by $j\omega$ is the same as differentiating the FRF.
4. Dividing a (velocity/force) FRF by $j\omega$ is the same as integrating the FRF.
5. Multiplying a (velocity/force) FRF by $j\omega$ is ***not the same*** as differentiating it.
6. Multiplying a (displacement/force) FRF by $(j\omega)^2$ is ***not the same*** as twice differentiating it.
7. Dividing an (acceleration/force) FRF by $j\omega$ is ***not the same*** as integrating it.
8. Dividing an (acceleration/force) FRF by $(j\omega)^2$ is ***not the same*** as twice integrating it.