Vibrant Technology, Inc.

ME'scopeVES Application Note #28

Mathematics of a Mass-Spring-Damper System

INTRODUCTION

In this note, the capabilities of ME'scopeVES will be used to build a model of the mass-spring-damper system shown in Figure 1. Then, its equation of motion will be solved for its mode of vibration. Then, an FRF will be synthesized using the modal data, and its stiffness and mass lines will be shown.

Then, the FRF will be curve fit to recover its modal and physical parameters. Finally, we will look at how the modal parameters are contained in the Impulse Response Function, the Inverse FFT of the FRF.

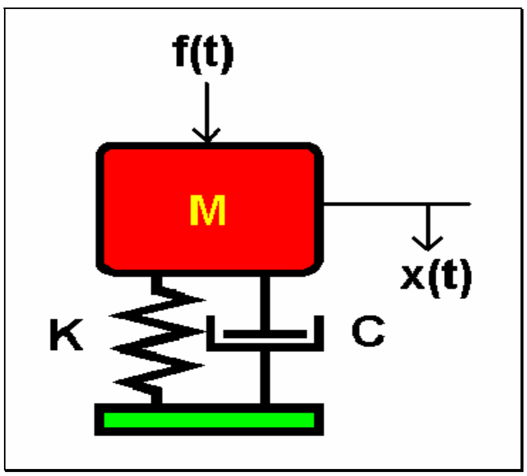


Figure 1. Mass-Spring-Damper.

The purpose of this Application Note is to review the details of modal analysis, to provide a better understanding of the modal properties of all structures. The modal properties of real world structures are analyzed using a multi-degree-of-freedom (MDOF) dynamic model, whereas the model used here is a single degree-of-freedom (SDOF) model. Nevertheless, the dynamics of MDOF structures are better understood by analyzing the dynamics of this SDOF structure.

Modes are defined for structures, the dynamics of which can be represented by linear ordinary differential equations like Equation 1. The dynamic behavior of the mass-spring-damper structure in Figure 1 is represented by a single (scalar) equation, Equation 1. (An MDOF structure is represented by multiple equations like Equation 1, which are written in matrix form.)

Because of the superposition property of linear systems, the dynamics of an MDOF structure can be written as a ***summation of contributions due to each of its modes***. Each mode can be thought of as representing the dynamics of a single mass-spring-damper system.

BACKGROUND MATH

The time domain equation of motion for the mass-spring-damper is represented by Newton's Second Law,

$$\mathbf{M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t)} \qquad (1)$$

where:

$\mathbf{M}$ = **mass value**

$\mathbf{C}$ = **damping coefficient**

$\mathbf{K}$ = **spring stiffness**

$\mathbf{\ddot{x}(t)}$ = **acceleration**

$\mathbf{\dot{x}(t)}$ = **velocity**

$\mathbf{x(t)}$ = **displacement**

$\mathbf{f(t)}$ = **excitation force**

Laplace Transforms

By taking Laplace transforms of the terms in Equation 1 and setting initial conditions to zero, an equivalent frequency domain equation of motion results,

$$\mathbf{[M\,s^2 + C\,s + K]\ X(s) = F(s)} \qquad (2)$$

where: $\mathbf{X(s)}$ = Laplace transform of the displacement
$\mathbf{F(s)}$ = Laplace transform of the force
$\mathbf{s = \sigma + j\omega}$ = complex Laplace variable

Transfer Function

Equation 2 can be rewritten by simply dividing both sides by the coefficients of the left-hand side.

$$\mathbf{X(s) = \left(\frac{1}{M\,s^2 + C\,s + K}\right) F(s)} \qquad (3)$$

The new coefficient on the right hand side of Equation 3 is called the Transfer Function**,**

$$\mathbf{H(s) = \frac{X(s)}{F(s)} = \left(\frac{1}{M\,s^2 + C\,s + K}\right)} \qquad (4)$$

The Transfer Function is complex valued, and therefore has two parts; ***real & imaginary*** or equivalently ***magnitude & phase***. The two parts of the Transfer Function can be plotted on the complex Laplace plane (or **s**-plane), as shown in Figure 2.

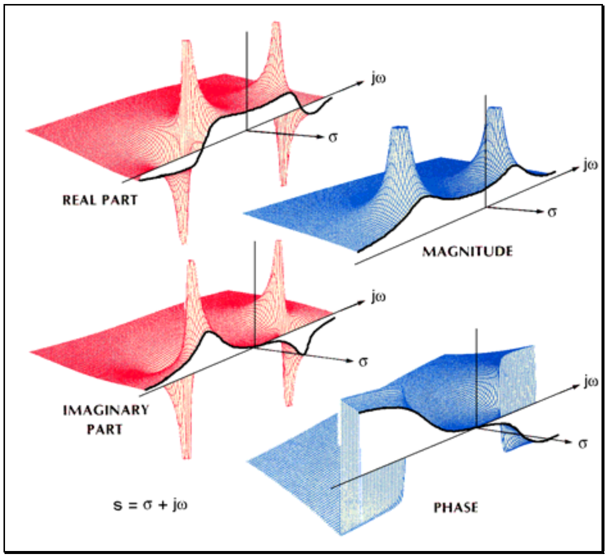


Figure 2. Transfer Function on the S-Plane.

Poles of the Transfer Function

Notice that the magnitude of the Transfer Function has two peaks in it. These are points where the value of the Transfer Function ***goes to infinity***. (The real and imaginary parts also show the same two peaks.)

By inspection of Equation 4, it is clear that the Transfer Function goes to infinity for values on the **s**-Plane where its ***denominator is zero***. It is also clear that as **s** goes to infinity, the Transfer Function will go to zero.

The denominator is a second order polynomial in the **s** variable, called the ***characteristic polynomial***. Since it is a second order, it has two roots (values of **s**) for which it will be zero. These two roots of the denominator are called the ***poles*** of the Transfer Function. Furthermore, the poles are complex conjugates of one another. The poles therefore, are the locations in the **s**-plane where the Transfer Function has a value of infinity. The poles are also called **eigenvalues**.

$$\mathbf{p_0 = -\sigma_0 + j\omega_0}\ , \quad \mathbf{p_0{*} = -\sigma_0 - j\omega_0} \qquad (5)$$

S-Plane Nomenclature

The real axis in the **s**-Plane is called the ***damping axis***, and the imaginary axis is called the ***frequency axis***. The locations of the poles in the **s**-Plane have also been given some other commonly used names, as shown in Figure 3.

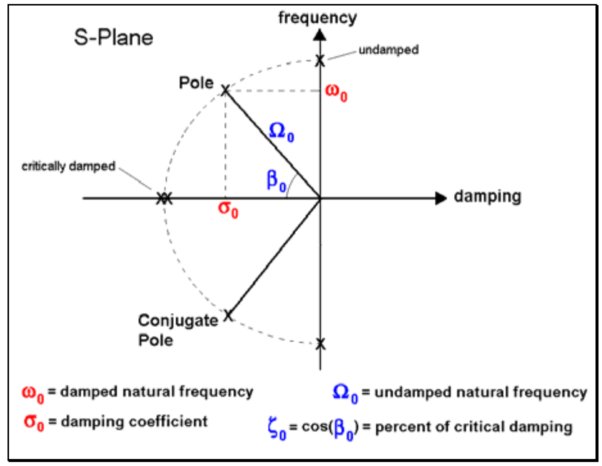


Figure 3. S-Plane Nomenclature.

Modal Parameters

The coordinates of the poles in the **s**-Plane are also modal parameters. Rewriting Equation 4 in terms of its pole locations, or modal parameters,

$$\mathbf{H(s)} = \left(\frac{\mathbf{1/M}}{\mathbf{s^2 + 2\sigma_0 s + \Omega_0{}^2}} \right) \qquad (6)$$

$$\text{where:} \quad \sigma_0 = \frac{\mathbf{C}}{\mathbf{2\,M}}\ , \quad \Omega_0{}^2 = \frac{\mathbf{K}}{\mathbf{M}} \qquad (7)$$

$$\sigma_0 = \textbf{modal damping coefficient}$$

$$\Omega_0 = \textbf{undamped modal frequency}$$

$$\Omega_0{}^2 = \sigma_0{}^2 + \omega_0{}^2$$

$$\omega_0 = \textbf{damped modal frequency}$$

And, the ***percent of critical damping*** (ζ_0) is written as,

$$\zeta_0 = \frac{\sigma_0}{\Omega_0} = \frac{\mathbf{C}}{\mathbf{2\sqrt{MK}}} \qquad (8)$$

Frequency Response Function (FRF)

Notice that in Figure 2 ***the Transfer Function has only been plotted for half of the s-Plane.*** That is, it has only been plotted for negative values of σ (the real part of **s**). This was done so that the values of the Transfer Function along the **j**ω-axis (the imaginary part of **s**) are clearly seen.

Definition: The **Frequency Response Function** (or **FRF**) is the values of the Transfer Function along the **jω**-axis.

The FRF values are pointed out in Figure 4. Since the FRF is only defined along the **jω**-axis, **s** can be expressed in terms of **jω,**

$$\mathbf{FRF} = \mathbf{H(j\omega)} = \mathbf{H(s)}\Big|_{s=j\omega}$$

$$= \frac{\mathbf{X(s)}}{\mathbf{F(s)}}\Bigg|_{s=j\omega} = \frac{\mathbf{X(j\omega)}}{\mathbf{F(j\omega)}} \qquad (9)$$

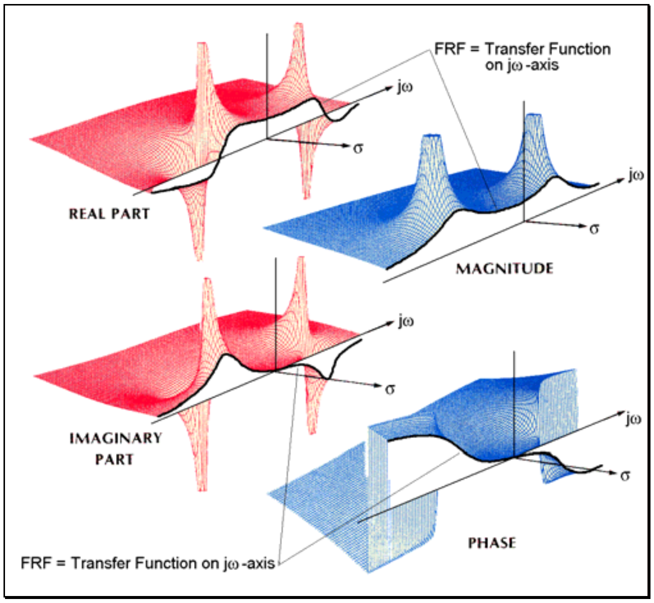


Figure 4. FRF Plotted on the jω-axis.

FRF in Partial Fraction Form

The FRF for an SDOF can now be written by simply replacing the **s**-Variable in Equation 6 with **jω**,

$$\mathbf{H(j\omega)} = \frac{(\mathbf{1/M})}{(\mathbf{j\omega})^2 + \mathbf{2\sigma_0\ j\omega} + \mathbf{\Omega_0}^2} \qquad (10)$$

Furthermore, using the poles of the Transfer Function, a partial fraction expansion can be performed on Equation 10 yielding,

$$\mathbf{H(j\omega)} = \frac{\mathbf{1}}{\mathbf{2j}}\left[\frac{\mathbf{R_0}}{\mathbf{j\omega - p_0}} - \frac{\mathbf{R_0}}{\mathbf{j\omega - p_0\,*}}\right] \qquad (11)$$

where: $\mathbf{R_0 = 1/\omega_0 M}$ (12)

$\mathbf{R_0}$ is called the modal ***residue***. It is simply the amplitude (or strength) of the numerator of each resonance term in Equation 11. Comparing Equation 11 with the FRF in Figure 4, it is clear that the FRF of an SDOF is the ***summation of two resonance curves***, each one forming a peak near one of the two pole locations,

$$\mathbf{p_0 = -\sigma_0 + j\omega_0}\ \ \&\ \ \mathbf{p_0{*} = -\sigma_0 - j\omega_0}.$$

Equation 11 also says that the FRF for an SDOF is ***fully represented by two poles and two residues***. Furthermore, since the residues are equal and the poles are complex conjugates of one another, the complete dynamics of an SDOF system is ***fully represented by a modal frequency*** $(\mathbf{j\omega_0})$, ***modal damping*** $(\mathbf{\sigma_0})$, and ***modal residue*** $(\mathbf{R_0})$.

Mode Shapes

One final step is to represent the FRF in terms of mode shapes instead of residues,

$$\mathbf{H(j\omega)} = \frac{\mathbf{1}}{\mathbf{2j}}\left[\frac{\{\mathbf{u_0}\}^2}{\mathbf{j\omega - p_0}} - \frac{\{\mathbf{u_0}\}^2}{\mathbf{j\omega - p_0\,*}}\right] \qquad (13)$$

where:

$$\{\mathbf{u_0}\} = \left\{\begin{matrix}\dfrac{\mathbf{1}}{\sqrt{\mathbf{A\omega_0 M}}} \\ \mathbf{0}\end{matrix}\right\},\ \mathbf{A = scaling\ constant} \qquad (14)$$

Notice that the mode shape $\{\mathbf{u_0}\}$ is a vector. Its first component is the square root of the residue, and its second component is zero. The second component corresponds to the ***ground***, where there is no motion. Notice also that the mode shape contains a scaling constant (**A**). This is because a ***mode shape doesn't have unique values. Only its shape (one component relative to another) is unique***. A mode shape is also called an ***eigenvector***.

Equation 13 states that the dynamics of an SDOF is fully represented by a ***pair of eigenvalues (poles)*** and a ***pair of eigenvectors (mode shapes)***. Because of symmetry, Equation 13 also says that the dynamics is fully represented by a ***modal frequency*** $(\mathbf{j\omega_0})$, ***modal damping*** $(\mathbf{\sigma_0})$, and a ***mode shape*** $\{\mathbf{u_0}\}$.

Impulse Response Function (IRF)

The Impulse Response Function is the Inverse FFT of the FRF. It too can be written in terms of modal parameters and provides the best source of meaning for the modal parameters.

Using Equation 11 to express the FRF in terms of modal parameters, the IRF is written,

$$\mathbf{h(t) = FFT^{-1}\left(\frac{1}{2j}\left[\frac{R_0}{j\omega - p_0} - \frac{R_0}{j\omega - p_0 *}\right]\right)} \qquad (15)$$

or

$$\mathbf{h(t) = \frac{T}{2j}\left[R_0 e^{p_0 t} - R_0 e^{p_0 * t}\right]} \qquad (16)$$

or

$$\mathbf{h(t) = T\,|R_0|e^{-\sigma_0 t}\left(\sin(\omega_0 t + \alpha_0)\right)} \qquad (17)$$

where: $\mathbf{\alpha_0 = angle\ of\ R_0}$

Equation 17 shows clearly the role that each modal parameter plays in the IRF. ω_0 multiplies the time variable (**t**) in the sinusoidal function $\left(\sin(\omega_0 t + \alpha_0)\right)$. It defines the frequency of oscillation, hence ω_0 is called ***modal frequency***. σ_0 is the coefficient in the exponential term $\left(e^{-\sigma_0 t}\right)$ that defines the envelope of decay for the IRF. Since the decay is caused by a combination of damping mechanisms within or about a structure, σ_0 is called the ***modal damping***, or damping coefficient.

A NUMERICAL EXAMPLE

We will build a dynamic model for a mass-spring-damper system using ME'scopeVES. The ***Visual* Modal, Modal Pro** or ***Visual* SDM** option is required to carry out all steps of this exercise. If you have the ***Visual* SDM** option, you can add mass, stiffness, and damping elements directly to the SDOF model and generate its modal parameters. Otherwise, you will have to enter the modal parameters into a Shape Table.

Building the 3D Model

- Start a new Project in ME'scopeVES by executing **File | Project | New**.

The mass, spring, & damper element values must have units associated with them. Both English and Metric units are included below. Take your pick.

- Execute **File | Options** in the Structure window, and select the following units.

Metric Units: Mass Units = Kilograms (kg)
Force Units = Newtons (N)
Length Units = Meters (m)

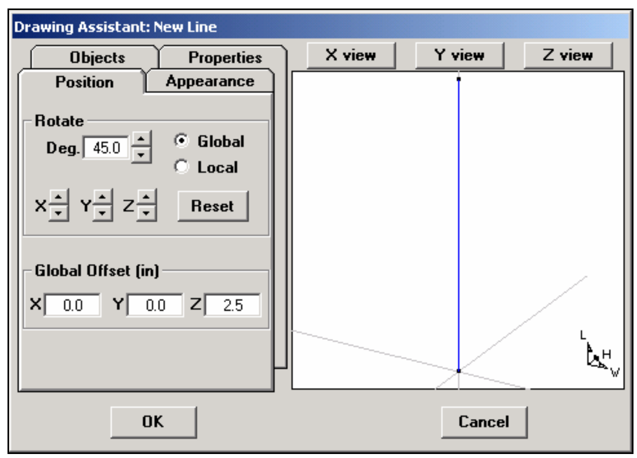
English Units: Mass Units = Pounds (lbm)
Force Units = Pounds-force (lbf)
Length Units = Inches (in)

- While the Structure Option box is open, click on the **Labels & Grids** tab, and check **Show Axis Lines**.

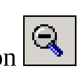
The 3D model will consist of essentially two points, one for the mass and one for ground. Later, we will also draw a box around the mass point, and a rectangle around the ground point to make them look more realistic.

To create a SubStructure made up of two points connected by a line,

- Open the Drawing Assistant Box by executing **Draw | Drawing Assistant**.
- On the **Objects** tab, **double click** on the **Line** Object. A **Line** SubStructure will be visible in the View on the right.
- On the **Properties** tab, set the **Length** parameter to **5** and set the **Points** parameter to **2**.
- On the **Position** Tab, to rotate the Line to the vertical position, select **Local** coordinates, enter **45** in the **Rotate Deg.** box, and click on **W down arrow *twice***.
- To move the Line to by proper location, enter a **Global Offset** of **X=0, Y=0, Z=2.5,** and press **OK**.



The new Substructure (a vertical Line) will be displayed in the Structure window.

- To display the two endpoints of the Line, execute **Display | Points**.
- Press the mooZ button  to make the Points clearly visible.

To label the Points with numbers,

- Execute **Draw | Points | Number Points**. A dialog box will open.

- Click near the **top** Point to number it **1**, and then near the **bottom** Point to number it **2**.
- Click on **Done** in the dialog box to close it.

Adding the Mass Cube

To add the cube centered at the mass (top) point,

- Open the Drawing Assistant again by executing **Draw | Drawing Assistant**.
- On the **Objects** tab, **double click** on the **Cube** Object to place a **Cube** SubStructure in the View on the right side.

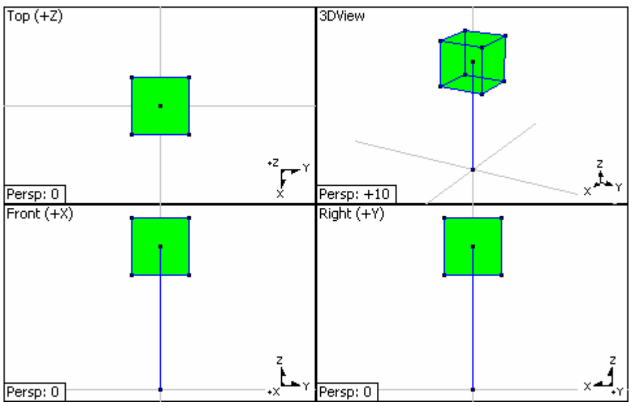


Figure 5. Line & Cube.

- On the **Properties** tab, change the **Length, Width** & **Height** parameters to **2** and change the **Points** parameters to **2**.
- Enter a **Global Offset** of **X=0, Y=0, Z=5,** and press **OK**. The Structure should now look like the one in Figure 5.

Adding the Ground Plane

The ground plane will be modeled with a square plate SubStructure.

- Execute **Draw | Drawing Assistant** again.
- On the **Objects** tab, **double click** on the **Plate** Object. This will place a **Plate** SubStructure in the View on the right.
- On the **Properties** tab, change the **Length** & **Width** parameters to **3,** and change their **Points** parameters to **2**.
- Set the **Height** and the number of Points in the Height direction to **0**.
- Enter a **Global Offset** of **X=0, Y=0, Z=0,** and press **OK**.

The Structure should now look like that in Figure 6.

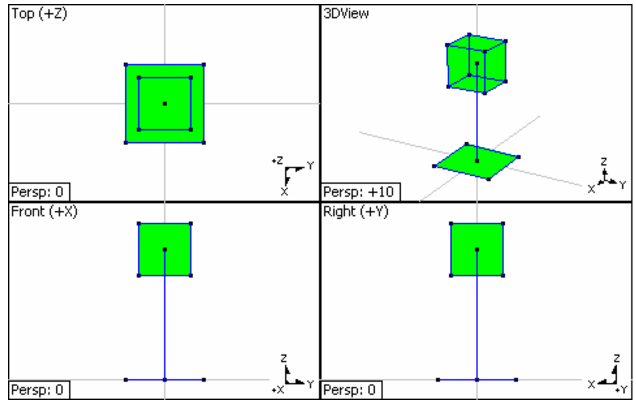


Figure 6. Line, Cube & Plate.

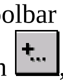
Creating an Undamped Mode

First, we will create an undamped **10 Hz mode** of a mass on a spring. With the *Visual* SDM option, mass and stiffness elements are added directly to the 3D model. Then, the undamped mode of vibration can be calculated by executing **Modify | Calculate Element Modes**. (If you don't have the *Visual* SDM option, skip to the section called "Entering the Undamped Mode into a Shape Table**"**.)

We will choose a mass value, and the spring stiffness will be computed using Equation 7. The amount of mass depends on the mass units chosen.

Metric Units:	1 N = 1 kg x 1(m/sec^2), or 1 kg = 1 N-sec^2/m
English Units:	1 lbf = 1 lbm x 386.09 (in/sec^2), or 386.09 lbm = 1 lbf-sec^2/in

Depending on the units you chose, either a **1 kg mass** or a **386.09 lbm mass** will be used.

- To add the mass element, select **FE Masses** from the Object list on the Toolbar in the Structure window, press the **Add** button , and click near **Point 1** (in the center of the cube).

This will add the mass element to the model and create a new row in the mass properties spreadsheet.

- Drag the **vertical blue bar** to the left, and enter the mass value (either **386.09 lbm** or **1 kg**) into the spreadsheet.
- Select **Global Z** in the **Orientation** column in the masses spreadsheet.

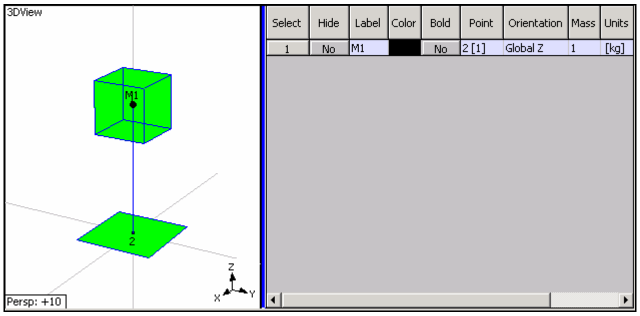


Figure 7. Mass Element.

To create a 10 Hz undamped mode, Equation 7 can be used to calculate the spring stiffness,

$$K = M\,\Omega_0^{\,2} = \mathbf{1} \text{ x } (\mathbf{10}\text{Hz})^2 = (\mathbf{20}\pi)^2 = \mathbf{3947.8}$$

The stiffness units are either (**lbf/in**) or (**N/m**).

- To add the spring to the model, select **FE Springs** from the Object list, press the **Add** button , and click on **Point 1,** and then on **Point 2**.

This will add a spring element to the model, and create a new row in the spring properties spreadsheet.

- Enter the stiffness value (**3947.8**) into the Stiffness cell in the spreadsheet.

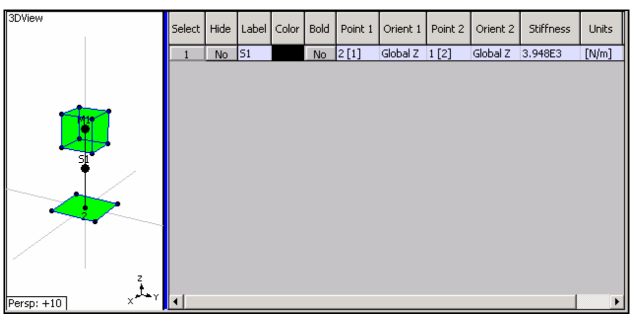
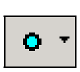


Figure 8. Spring Element.

- Select **Global Z** from both the **Orient 1** and **Orient 2** columns in the spreadsheet.

Fixing the Ground Point

Finally, the ground point needs to be ***defined as ground*** by making its motion Fixed. (A Fixed Point has no motion.) To select **Point 2** (and the points on the square plate),

- Select **Points**  from the Object list on the Toolbar. Hold down the **Alt** key, and draw a **Selection Box** around the **ground point** and the **square plate**.
- Execute **Draw | Points | Fixed** to fix the selected points in the X, Y and Z directions.

Calculating the Modal Parameters

Now the model is ready to calculate the undamped mode of vibration.

- Execute **Modify | Calculate Element Modes**, and click on **OK** to accept the default Shape Table name.

The Shape Table will now open with the undamped 10 Hz mode in it.

- Execute **Display | Shapes** to view the Shapes spreadsheet.

Notice in Figure 9, that the measurement type is **Unit Modal Mass mode shape**, and that its magnitude value is **1**.

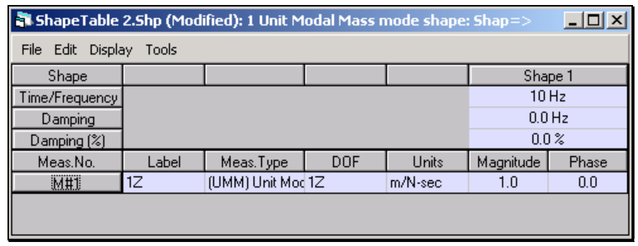


Figure 9. Undamped Mode Shape.

Entering the Undamped Mode into a Shape Table (*Visual* Modal & Modal Pro options)

If you skipped the previous sections, you can manually enter the modal parameters of the undamped mode directly into a Shape Table.

- Execute **File | New | Shape Table** from the ME'scopeVES window, click on **OK**, enter a **1** for number of DOFs, and click on **OK** to open the new Shape Table.
- To enter the modal frequency, select the **Frequency** cell and type **10.**

Mode Shape as Residue

To define the shape as a Residue mode shape,

- Click on the **Measurement Type** cell heading. A dialog box will open. Select **Residue Mode Shape** from the list, and click on **OK**.
- To define the DOFs of the shape, select the cell in the **DOF** column and type **1Z:1Z.**

From Equation 11, it is clear that the residue units are,

Residue units = (**FRF units**) x (**Radians/Second**)

This is because the units of the FRF denominator are radians/second (Hz, RPM or CPM). Therefore, the units are,

Metric Residue Units: (m/N-sec)

English Residue Units: (in/lbf-sec)

- To define the residue units, select the cell in the **Units** column and type either **m/N-sec** or **in/lbf-sec.**

The modal residue value is computed using Equation 12.

$$\mathbf{R_0 = 1/\omega_0 M = 1/20\pi = 0.015915}$$

- To enter the residue, select the cell in the **Magnitude** column, and type **0.015915**.

Scaling the Mode Shape to Unit Modal Mass

Even though the mode shape can be defined using residues, (which have unique values), the mode shape has no unique values. ***Its shape is unique, but its values are not***.

A common way to scale mode shapes is so that they yield unit modal masses. That is, if the mass is pre- and post-multiplied by the mode shape, the resulting ***modal mass*** will be ***unity***, or **1**. (For an MDOF system, pre- and post-multiplying the mass matrix by a mode shape vector will yield a modal mass of unity.)

To scale the mode shape to unit modal mass requires a specific value for the scaling constant (A) in Equation 14

$$\mathbf{\{u_0\}M\{u_0\} = 1 \quad => \quad A = \frac{1}{\omega_0}}$$

Using this value for the scaling constant (**A**) in Equation 14 gives a unit modal mass scaled mode shape,

$$\mathbf{\{u_0\} = \begin{Bmatrix} \frac{1}{\sqrt{M}} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}}$$

To scale the mode shape to unit modal masses,

- Execute **Display | Shapes** to show the shape.
- Execute **Tools | Scaling | Unit Modal Mass**.

Adding 5% Critical Damping to the Mode

To add damping to the mode, either a Damper can be added to the 3D model (using the *Visual* SDM option), or the damping can be entered directly into the Shape Table.

- If you don't have the *Visual* SDM option, select the **Damping %** cell in the Shape Table, and enter a **5**.

If you have the *Visual* SDM option, you can add a damper to the 3D model in the Structure window, and compute a new damped mode of vibration. Equation 8 is the relationship between Percent of Critical damping and the damper coefficient (**C**). Solving for this coefficient gives,

$$\mathbf{C = 2\zeta_0\sqrt{MK} = 2(0.05)\sqrt{(20\pi)^2} = 2\pi}$$

- To add the damper to the model, select **FE Dampers** from the Object list, press the **Add** button, and click on **Point 1,** and then **Point 2**.

This will add the damper element to the model, and create a new row in the damper properties spreadsheet, as shown below.

- Enter the damping value (**6.283**) into the spreadsheet.
- Select **Global Z** from both the **Orient 1** and **Orient 2** columns in the spreadsheet.

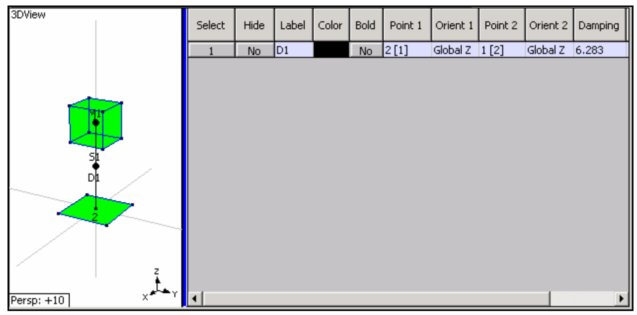


Figure 10. Damper Element.

To generate the damped mode of vibration,

- Execute **Modify | Calculate Element Modes** and click on **OK** to accept the new Shape Table name.

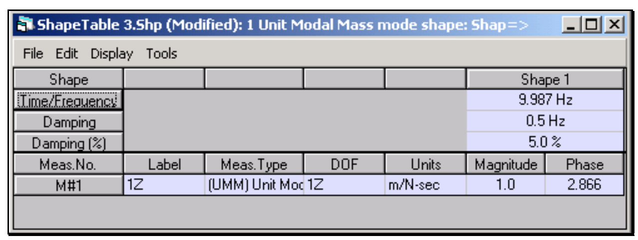


Figure 11. Damped Mode Shape.

The Shape Table will open with the damped mode in it. Notice that the damped modal frequency is now slightly less than 10 Hz. We can check this with the formula for damped natural frequency.

$$\sigma_0 = \frac{\mathbf{C}}{\mathbf{2M}} = \frac{2\pi}{\mathbf{2}} = \pi \quad \frac{\mathbf{rad}}{\text{sec}}$$

$$\omega_0 = \sqrt{\Omega_0^{\,2} - \sigma_0^{\,2}}$$
$$= \sqrt{(20\pi)^2 - \pi^2}$$
$$= \pi\sqrt{399}\ \frac{\text{rad}}{\text{sec}}$$
$$= 9.9875\ \text{Hz}$$

- If you don't have the *Visual* SDM option, change the modal Frequency to **9.9875** in the Shape Table.

Synthesizing an FRF Using Modal Parameters

The FRF, like a Transfer Function, defines the dynamic characteristics between two DOFs of a structure. MDOF systems have many DOF pairs for which FRFs can be derived or measured. The SDOF system in Figure 1 has only one meaningful DOF, motion at the mass point (**Point 1**) in the vertical (Z) direction. Therefore, using modal parameters we can synthesize an FRF between the DOF (**1Z**) and itself. ***Any FRF between a DOF and itself is called a driving point FRF***.

Since the FRF is a ratio of response (or output) over excitation (or input) its units are either (**in/lbf**) or (**m/N**). Its DOFs are denoted as **1Z : 1Z**.

NOTE: Equation 14 shows that the mode shape is real valued. Therefore, its phase should be set to zero (0) in the Shape Table window.

- Execute **Tools | Synthesize FRFs** in the Shape Table window. The FRF Synthesis dialog box will open, allowing you to edit the frequency axis parameters of the FRF, and select roving and reference DOFs for synthesis.

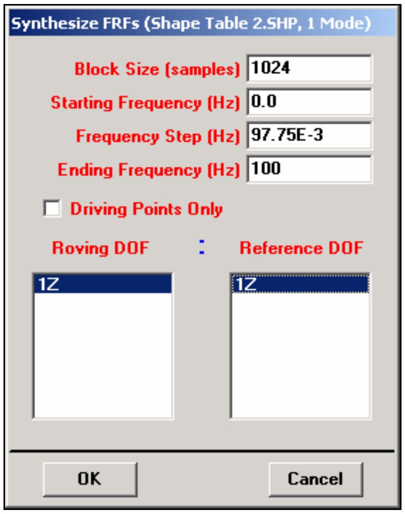


Figure 12. FRF Synthesis Dialog Box.

- Edit the ending frequency to **100** Hz, select the two DOFs, and click on **OK**.
- Click on **OK** again to accept the default Data Block name.
- Execute **Display | Bode | Upper/Lower** in the Data Block window to display the synthesized FRF.

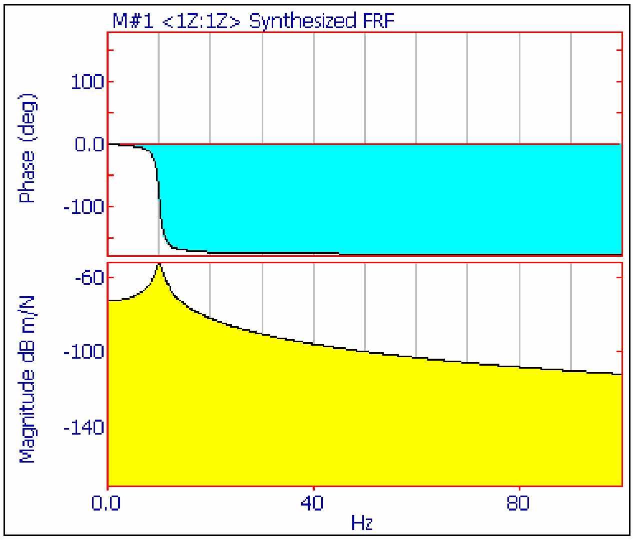


Figure 13. (Displacement/Force) FRF.

Stiffness & Mass Lines

We will now look for the stiffness and mass lines in the FRF. In the Data Block window containing the (displacement/force) FRF,

- Execute **Display | Magnitude.** Then, execute **Format | Vertical Axis Scaling** and choose **Log Magnitude** and **4 decades.**

- Execute **Format | Horizontal Axis Scaling** and choose **Log Frequency** to set the horizontal axis display scale to log.

Stiffness Line

As the frequency approaches **0 Hz**, Equation (4) shows that the (displacement/force) FRF approaches a constant, equal to the ***inverse of the stiffness***. (This is also called the ***flexibility***.) By letting **S=0** in Equation (4), the flexibility is,

$$\mathbf{H(0) \approx \frac{1}{K} = \frac{1}{3947.8} = 0.0002533}$$

- To check the synthesized FRF, turn on the Line cursor and move it to the left side of the Trace. The cursor value will be **approximately 0.000253**.

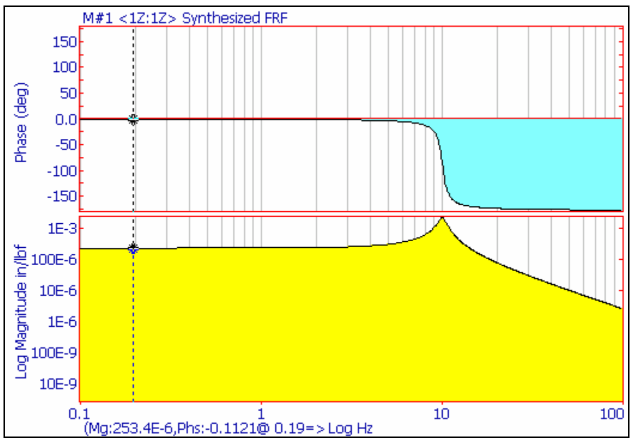


Figure 14. Stiffness Line.

Mass Line

As frequency becomes large, the FRF becomes dominated by the ω^2 term in its denominator (characteristic polynomial). (*See* Equation (10).) Therefore, at high frequencies the magnitude of the FRF can be approximated by,

$$\mathbf{\left|H(\omega)\right| \approx \left(\frac{1}{M\omega^2}\right)}$$

We can use the **Tools | Math | Differentiate** command ***twice*** to multiply the (displacement/force) FRF by ω^2. Hence, its magnitude for high frequencies will become a constant,

$$\mathbf{\omega^2 \left|H(\omega)\right| \approx \left(\frac{1}{M}\right)}$$

To check this,

- Execute **Tools | Math | Differentiate *twice*** in the Data Block window that showed the Stiffness line.
- Execute **Format | Horizontal Axis Spacing,** and select **Frequency** to change back to a linear Frequency Axis.
- Move the Line cursor to a high frequency on the Magnitude Trace.

The cursor value is ***approximately*** **1**, the mass of the structure.

NOTE: In **Application Note #16, Integration & Differentiation of FRFs**, it is shown that multiplying a (displacement/force) FRF by ω^2 ***does not*** yield the correct (acceleration/force) FRF. Nevertheless, it is still useful for finding the mass line of an SDOF system as done here.

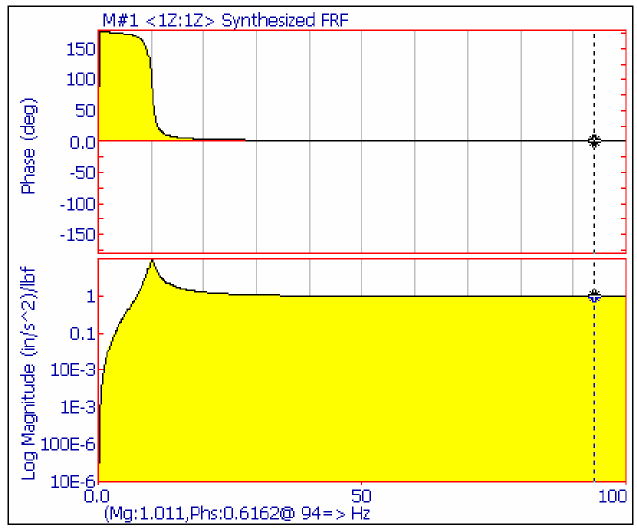


Figure 15. Mass Line.

Modal Parameter Estimation (Curve Fitting)

The FRF in Figure 13 is a complete representation of the dynamics of the mass-spring-damper system at its driving point. The FRF can be curve fit to recover the modal parameters of the SDOF, which are also a complete representation of the dynamics.

To curve fit the FRF in Figure 13,

- Execute **Modes | Modal Parameters** in the (displacement/force) FRF Data Block window. The Curve Fitting panel will be displayed.
- **Right click** on the one of the column headings in the modal parameters spreadsheet. The Options box will open showing the Curve Fitting tab.
- *Check* the **mass**, **damping** & **stiffness** columns to display them in the spreadsheet, and click on **OK**.
- Spread the edges of the Band cursor to enclose the modal peak at **10 Hz**.

- Select **Modal Peaks Function** and press the **Count Peaks** button.
- On the **2. Frequency and Damping** tab, select **Global Polynomial** and press the **F & D** button.
- On the **3. Residues & Save Shapes** tab, select **Global Polynomial** and press the **Residues** button.

The modal parameter estimates are displayed in the spread-sheet, as shown in Figure 16.

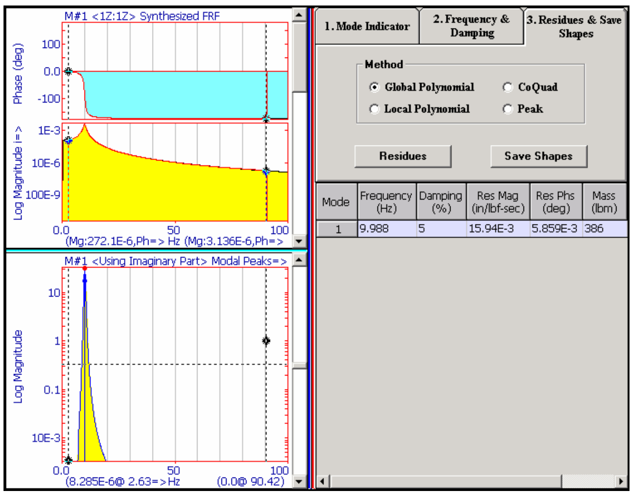


Figure 16. Curve Fitting Synthesized FRF.

Note that the modal **frequency**, **damping** & **residue** values, plus the **mass**, **damping** & **stiffness** values are all the correct values of the SDOF system.

Impulse Response Units

From Equation 17, it is clear that the Impulse Response is the product of two functions, an ***exponential*** function $(\mathbf{T}\,|\mathbf{R}_0|\,\mathbf{e}^{-\sigma_0 t})$ and a ***sinusoidal*** function $(\sin(\omega_0 t + \alpha))$.

The exponential decay $(\mathbf{e}^{-\sigma_0 t})$ and sinusoidal function are dimensionless and only **T** and $\mathbf{R}_0$ have units. Therefore,

Impulse Response units = (**Seconds**) x (**Residue units**)

or,

Impulse Response units = **FRF units**

Envelope of the Impulse Response

The maximum possible magnitude of the Impulse Response is the value of the exponential function for **t=0**,

$$\mathbf{max.\,mag.} = \mathbf{T}\,|\mathbf{R}_0|\,\mathbf{e}^{-\sigma_0 t}\Big|_{t=0} = \mathbf{T}\,|\mathbf{R}_0|$$

This is the product of the time length (**T**) of the Trace, and the magnitude of the residue $|\mathbf{R}_0|$.

To check the envelope of the Impulse Response, a new FRF will be synthesized using the modal parameters, but this time the frequency increment will be chosen so that **T=1**.

$$\Delta \mathbf{f} = \frac{\mathbf{1}}{\mathbf{T}} = \mathbf{1\ Hz}$$

- In the Shape Table with the damped mode in it, execute **Tools | Synthesize FRFs**. The FRF Synthesis dialog box will open.
- Change the **Frequency Increment** to **1**, select both Roving & Reference DOFs (**1Z)**, and click on **OK**.

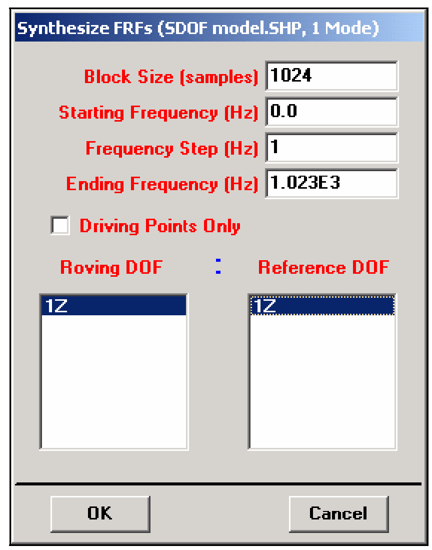


Figure 16. Synthesizing a New FRF for computing IRF.

- To obtain the Impulse Response, execute **Transform | Inverse FFT** in the newly synthesized Data Block.
- Turn on the **Line** cursor, and move it to first peak in the Impulse Response.

Notice that the peak value of the Impulse Response is **0.015**. Since **T=1**, the maximum envelope value (for t = 0) should be slightly larger, approximately equal to the ***magnitude of the residue,* 0.0159**.

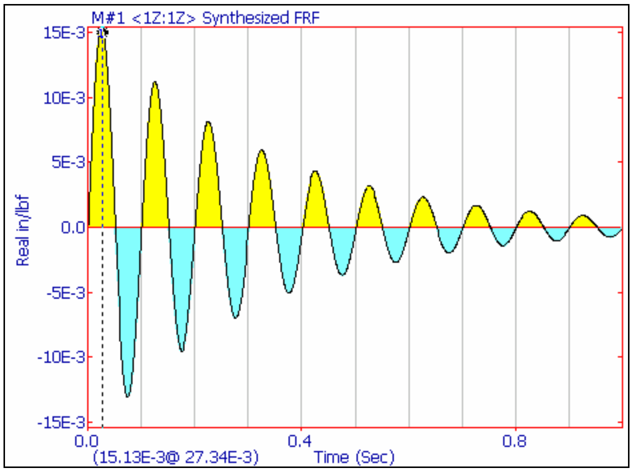


Figure 17. Impulse Response Function.